Abstract

This paper surveys three areas of applications of dynamic games: (i) transboundary pollution, (ii) exploitation of transboundary resources, and (iii) problems of the developing world: capital flights, foreign aid, war and peace. A refresher section introduces the basic ideas and solution techniques in differential games.

1 Introduction

One of the emerging trends in the expanding relationship between game theory and the social sciences is the application of dynamic/stochastic games in a variety of settings. The purpose of this paper is provide an introduction to the applications of dynamic games to global and transboundary issues. The focus will be on differential games, though timing games will also be mentioned.

While economic analyses containing elements of differential games can be traced back to Roos (1925,1927, Journal of Political Economy), it can be argued that the systematic ideas underlying differential games arose from Isaacs’s Rand Paper P 257, in 1951, inspired by von Neumann at Rand. As dynamic game approached its 60th anniversary, a number of events seem to have contributed to the celebration of the scientific origin of this body of work. Among these are (i) the first issue of the new journal, Dynamic Games and Applications, appeared in 2011, under the guidance of Georges Zaccour, (ii) two major survey articles on dynamic games appeared in 2010 and 2011, dealing respectively with the economics of pollution (Jorgensen et al. 2010) and natural resources (Long,
This paper covers three areas of applications of dynamic games: (i) trans-boundary pollution, (ii) exploitation of transboundary resources, and (iii) problems of the developing world: capital flights, foreign aid, war and peace. To begin with, a refresher section introduces the basic ideas and solution techniques in differential games.

2 Differential Games: A Refresher

Differential games are dynamic games with a key feature: the environment in which players interact changes over time, and its evolution is affected by the actions of the players. An obvious example is the global warming game: countries know that their environmental policies affect the rate of change of GHGs concentration. The environment of the game is represented by one or several state variables (for example the physical capital stocks, or a stock of pollution, or a biomass). The law of evolution for each state variable is described by a differential equation or a difference equation. (In the latter case, the games are sometimes called difference games. However, it has become a standard practice to use the term “differential games” for both differential games and difference games.) In what follows, results are presented mainly for the continuous-time case. It is straightforward to derived their counterparts in the discrete time case.

Differential games are also called state-space games. Players influence the evolution of the state variables by using their control variables. The game begins at time $0$ and ends at time $T$ (we call $T$ the time horizon, which may be finite or infinite).

A player can be a person, or an organization, such as a firm, a political party, a country. I will use the pronoun “it” to refer to a generic player.

2.1 Description of a typical differential game

A differential game normally displays the following properties. First, the players receive a net benefit at every point of time. Second, the payoff for a player is the integral of its discounted net benefit over the time horizon, plus a terminal payment at time $T$ called the “scrap value” which depends on the value of the state variables at $T$. When agents optimize at some time $t > 0$, they take into account the stream of present and future net benefits, but the payoffs they have received prior to $t$ are no longer relevant. Third, the net benefit that a player receives at $t$ may depend both on the actions taken at $t$ and on the “state of the system” in that period, as represented by the state variables. Fourth, the state of the system changes over time, and the rate of change of the state

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1Dojoloso is the nick name for the book Differential Games and Applications in Economics, by Dockner, Jorgensen, Long, and Sorger (hence the four o’s in Dojoloso).
variables may depend on the actions of the players, as represented by their control variables. Fifth, the rate of change of each state variable is described as a differential equation.

Below is a description of a differential game in continuous time. (For a more formal description, see Dockner et al. 2000.) The game starts at time zero and ends at time $T$. There are $N$ players, and $n$ state variables, denoted by $x_i$ where $i = 1, 2, ..., n$. The vector of state variables is $x = (x_1, x_2, ..., x_n) \in \mathbf{X} \subseteq \mathbb{R}^n$. The set $\mathbf{S} \equiv \mathbf{X} \times [0, T]$ is called the state-date space. An element $(x, t)$ is called a (state, date) pair. Player $i$ has a vector of $m$ control variables, denoted by $u_i(t)$. It is required that $u_i(t) \in U_i \subseteq \mathbb{R}^m$. We call $U_i$ player $i$’s control space. The control space is $\mathbf{U} \equiv \prod_{i=1}^n U_i$.

The evolution of the system is described by a system of $n$ differential equations,

$$\dot{x}_k(t) = G_k(x(t), u_1(t), u_2(t), ..., u_N(t), t), \quad k = 1, 2, ..., n$$

where $x_k(0) = x_{k0}$, a given number. In vector notation,

$$\dot{x}(t) = G(x(t), u_1(t), u_2(t), ..., u_N(t), t)$$

Player $i$’s net benefit at time $t$ is

$$B_i(t) = B_i(x(t), u_1(t), u_2(t), ..., u_N(t), t)$$

where $B_i(.)$ is a differentiable function. The payoff of player $i$ is

$$\int_0^T e^{-r_1t} B_i(x, u_1, u_2, ..., u_N, t) dt + e^{-r_1T} S_i(x_T, T)$$

where $S_i(x_T, T)$ is its “scrap value function” and $r_1$ is a non-negative constant, called the discount rate of player $i$.

Each player $j$ seeks to maximize its payoff. A Nash equilibrium is a strategy profile such that each player’s strategy maximizes its payoff given the strategies of the other players. The main types of strategies will be outlined below.

The discrete-time counterpart of the above game involves the system of $n$ difference equations

$$x_k(t+1) = x_k(t) + G_k(x(t), u_1(t), u_2(t), ..., u_N(t), t) \equiv h_k(x(t), u_1(t), u_2(t), ..., u_N(t), t)$$

or, in vector notation,

$$\mathbf{x}(t+1) = \mathbf{h}(\mathbf{x}(t), u_1(t), u_2(t), ..., u_N(t), t) \equiv \mathbf{h}(\mathbf{x}(t), \mathbf{u}(t), t)$$

and the objective function for each player $i$,

$$\sum_{t=1}^T \delta_i^{t-1} B_i(x, u_1, u_2, ..., u_N, t) + \delta_i^T S_i(x_{T+1}, T + 1)$$

where $\delta_i \in [0, 1]$ is player $i$’s discount factor.

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2In what follows, the time argument $t$ will be suppressed when there is no risk of confusion.
2.2 Open-loop strategies and open-loop Nash equilibrium

If at the beginning of the game (at time zero) a player chooses a time-path for his control variables and is committed never to deviate from that time-path, we say that it uses an open-loop strategy. More formally, an open-loop strategy \( f_i \) is a (piece-wise continuous) function that determines player \( i \)'s actions at each time \( t \) as a function of \( t \) only. Thus

\[
f_i : [0, T] \rightarrow U_i \subseteq \mathbb{R}^m.
\]

The value taken by the control vector \( u_i \) at time \( t \) is determined by evaluating the function \( f_i \) at time \( t \):

\[
u_i(t) = f_i(t)
\]

Let \( F_j \) be the set of open-loop strategies that are available to player \( j \). The open-loop strategy space of the game is

\[
F = \prod_{i=1}^{N} F_i
\]

Once all the players have chosen their open-loop strategies, the evolution of the state variables is described by

\[
\dot{x}_k(t) = G_k(x(t), f_1(t), f_2(t), ..., f_N(t), t), \quad k = 1, 2, ..., n, \quad x_k(0) = x_{k0}
\]

or, in vector notation,

\[
\dot{x}(t) = G(x(t), f(t), t), \quad x(0) = x_0
\]

Assume that given the initial condition \( x_0 \), the above system of differential equations has a unique solution \( x^*(t) \). The payoff for player \( i \) is then

\[
R_i(x_0, f) = \int_0^T e^{-r_i t} B_i(x^*(t), f(t), t) dt + e^{-r_i T} S_i(x_T^*, T)
\]

An open-loop Nash equilibrium (OLNE) is defined as a strategy profile \( \hat{f} = (\hat{f}_1, \hat{f}_2, ..., \hat{f}_N) \in F \) such that no player can achieve a higher payoff by choosing a different open-loop strategy. Thus

\[
R_i(x_0, \hat{f}) \geq R_i(x_0, f_i, \hat{f}_{-i}) \text{ for all } i \text{ and all } f_i \in F_i.
\]

To find an open-loop Nash equilibrium, one uses the Maximum Principle (see, for example, Leonard and Long, 1992) to derive the necessary conditions of each player's optimal control problem, taking as given the time path of the control variables of all other players. Then one finds a fixed point \( \hat{f} \) such that all the necessary conditions for all players are satisfied. Next one verifies that the sufficient conditions are satisfied at that fixed point.

Example 1: Finding an open-loop Nash equilibrium in a simple dynamic game in discrete time
The following example is a prototype model of transboundary pollution. The players are Country $\alpha$ and Country $\beta$. The actions take place in period 1 and period 2 only. Let $u_i^t$ be the emission level of Country $i$ ($i = \alpha, \beta$) in period $t$, where $t = 1, 2$. The state variable is the stock of pollution, denoted by $x_t$. Assume that $x_1$ is given, and

$$x_{t+1} = x_t + u_\alpha^t + u_\beta^t$$

Country $i$’s net benefit in period $t$ is

$$B_i^t = A^t u_i^t - \frac{1}{2} (u_i^t)^2 - \frac{1}{2} x_t^2$$

for $t = 1, 2$ where $A^t$ is a positive parameter. The scrap value function is

$$S^i = -\frac{1}{2} x_3^2$$

The rate of discount is zero for simplicity.

The payoff of country $\alpha$ is then

$$R^\alpha = \sum_{t=1}^{2} \left[ A^\alpha u_\alpha^t - \frac{1}{2} (u_\alpha^t)^2 - \frac{1}{2} x_t^2 \right] - \frac{1}{2} x_3^2$$

Each country $i$ seeks an open-loop strategy, which is its time path of emission levels $(u_1^t, u_2^t)$, to maximize its payoff, taking as given the open-loop strategy of the other country.

Maximizing $R^\alpha$ with respect to $(u_1^\alpha, u_2^\alpha)$ yields two first order conditions. A similar set of first order conditions applies to Country $\beta$.

Solving this system of four first order conditions, we obtain the unique open-loop Nash equilibrium strategy profile:

$$u_1^\alpha = \frac{6}{11} A^\alpha - \frac{4}{11} x_1 - \frac{5}{11} A^\beta$$

$$u_2^\alpha = \frac{7}{11} A^\alpha - \frac{4}{11} x_1 - \frac{4}{11} A^\beta$$

$$u_1^\beta = \frac{6}{11} A^\beta - \frac{4}{11} x_1 - \frac{5}{11} A^\alpha$$

$$u_2^\beta = \frac{7}{11} A^\beta - \frac{4}{11} x_1 - \frac{4}{11} A^\alpha$$

It follows that the time path of the state variable under the OLNE is \{ $x_1, x_2, x_3$ \} where

$$x_2 = \frac{1}{11} (x_1 + A^\alpha + A^\beta)$$

$$x_3 = \frac{1}{11} (x_1 + 4A^\alpha + 4A^\beta)$$
The Nash equilibrium payoff of each country can then be calculated. It is easy to show that the Nash equilibrium payoff pair is not Pareto efficient: both countries can better off if they cooperate.

**Time-consistency:** Open-loop Nash equilibria are time-consistent. This means that if any time \( t > 0 \) a player re-examines its strategies, it will find it optimal to continue with its originally chose path, as long as no other players have deviated from their equilibrium paths. This time-consistency property follows directly from the fact that a player’s original choice of strategy obeys Bellman’s Principle of Optimality.\(^3\)

On the other hand, if by mistake some players have deviated from his planned course of action, so that the stock size \( x(t_1) \) is different from what was anticipated at time zero, then at \( t_1 \) the players will in general find that they would be better off by switching to another open-loop strategy. We conclude that open-loop Nash equilibria are not robust to “trembling hand” deviations (Selten, 1975). One may say that open-loop Nash equilibria are not “subgame perfect”. For this reason, we now turn to an alternative equilibrium concept, the Markov-perfect Nash equilibrium, which satisfies the subgame perfect property.

### 2.3 Markovian strategies and Markov-perfect Nash equilibrium

We define a Markovian strategy for player \( j \) as a function that determines at each (state, date) pair what action to take. Let \( \phi_j : \mathbb{X} \times [0, T] \rightarrow \mathbb{U}_j \) be player \( j \)'s Markovian strategy, then

\[
\mathbf{u}_j(t) = \phi_j(\mathbf{x}(t), t)
\]

Let \( Q_j \) be the set of Markovian strategies that are available to player \( j \). Let \( Q = \prod_j Q_j \). Once all players have chosen their Markovian strategies, the evolution of the state variables is described by

\[
\dot{x}_i(t) = G_i(x(t), \phi_1(x(t), t), ..., \phi_N(x(t), t), t), \quad i = 1, 2, ..., n
\]

or, in vector notation,

\[
\dot{x}(t) = G(x(t), \phi(x(t), t), t)
\]

We assume the strategies are such that the above differential equation has a unique solution for any initial condition \((x_{t_1}, t_1)\). Let us define the performance index for player \( j \) at the (state, date) pair \((x, t)\) by

\[
J_j(x, t, \phi) = \int_t^T e^{-r_j(t-\tau)} B_j(x(\tau), \phi(x(\tau), \tau), \tau)) d\tau + e^{-r_j(T-t)} S_j(x_T, T)
\]

\(^3\)See e.g. Leonard and Long, 1992, Chapter 5 for a brief introduction to the Principle of Optimality.
We define a Markovian Nash equilibrium for a given \((x_0, 0)\) as a strategy profile \(\phi = (\phi_1, \phi_1, \ldots, \phi_1) \in Q\) such that

\[
J_j(x_0, 0, \phi) \geq J_j(x_0, 0, \phi_j, \phi_{-j}) \text{ for all } \phi_j \in Q_j
\]

A more demanding equilibrium concept is Markov-perfect Nash equilibrium (also called a feedback Nash equilibrium). It is defined as a strategy profile \(\phi = (\phi_1, \phi_2, \ldots, \phi_N) \in Q\) such that, at any (state, date) pair \((x, t) \in X \times [0, T]\), no player can make himself better off by choosing a different strategy, i.e., we require that, for each player \(j\),

\[
J_j(x, t, \phi) \geq J_j(x, t, \phi_j, \phi_{-j}) \text{ for all } \phi_j \in Q_j \text{ and for all } (x, t) \in X \times [0, T].
\]

It is important to stress that the concept is Markov-perfect Nash equilibrium requires that this inequality be satisfied for all possible (state, date) pairs \((x, t) \in X \times [0, T]\), not just for the initial pair at time zero, \((x_0, 0)\). As Reinganum and Stokey (1985) point out, a Markovian Nash equilibrium for a given \((x_0, 0)\) is not necessarily Markov-perfect. To be Markov-perfect, a Markovian Nash equilibrium must satisfy the additional property that the continuation of the given decision rules constitutes a Nash equilibrium when viewed from any future (date, state) pair. Dockner et al. (2000, example 4.2) give an example of a Markovian Nash equilibrium in decision rules that fails to be Markov-perfect. Maskin and Tirole (2001) explain why Markov-perfect Nash equilibrium is a useful equilibrium concept.

To find a Markov-perfect Nash equilibrium (MPNE) one makes use of the Hamilton-Jacobi-Bellman (HJB) equations. Let \(V_j(x, t)\) denote the value function for player \(j\). The HJB equation for player \(j\) is

\[
r_jV_j(x, t) - \frac{\partial V_j(x, t)}{\partial t} = \max_{u_j} \left\{ B_j(x, u_j, \phi_{-j}(x, t), t) + \frac{\partial V_j(x, t)}{\partial x} G(x, u_j, \phi_{-j}(x, t), t) \right\}
\]

with the boundary condition

\[
V_j(x, T) = S_j(x, T)
\]

If \(T\) is infinite, the boundary condition is replaced by

\[
\lim_{t \to \infty} e^{-r_j t} V_j(x(t), t) = 0.
\]

The discrete-time counterpart of the HJB equation is the Bellman equation

\[
W_j(x, t) = \max_{u_j} \left\{ B_j(x, u_j, \phi_{-j}(x), t) + \beta_j W_j \left( h(x, u_j, \phi_{-j}(x), t), t+1 \right) \right\}
\]

where \(t = 1, 2, \ldots, T\) (where \(\beta_j\) is player \(j\)'s discount factor). The boundary condition is

\[
W_j(x_{T+1}, T+1) = S_j(x_{T+1}, T+1).
\]

Remarks on stationary Markov-perfect strategies
If the functions \( B_j \) and \( G \) do not contain \( t \) as an argument, and if the time horizon is infinite, then at any time \( t \), each player faces the same optimization problem (except for the starting value of the state variables). It is then natural to look for decision rules that are time-independent, i.e. \( u_i(t) = \phi_i(x(t)) \). In this case, the HJB equation becomes

\[
 r_j V_j(x) = \max_{u_j} \left\{ B_j(x, u_j, \phi_{-j}(x)) + \frac{\partial V_j(x)}{\partial x} G(x, u_j, \phi_{-j}(x)) \right\}
\]

and, in the discrete-time case, the Bellman equation becomes

\[
 W_j(x) = \max_{u_j} \left\{ B_j(x, u_j, \phi_{-j}(x)) + \beta_j W_j \left( h(x, u_j, \phi_{-j}(x)) \right) \right\}
\]

**Example 1(b): Finding a Markov-perfect Nash equilibrium in a simple dynamic game in discrete time with a finite horizon**

Let us resume example 1 above, and suppose that the two countries use feedback strategies. We solve the game backwards. At the beginning of the last period, period 2, the two countries face a given pollution stock \( x_2 \). Let us find the Nash equilibrium for the last period, \( t = 2 \). Country \( \alpha \) chooses its emission level \( u_2^\alpha \) to maximize

\[
 A^\alpha u_2^\alpha - \frac{1}{2}(u_2^\alpha)^2 - \frac{1}{2} \left( x_2 + u_2^\alpha + u_2^\beta \right)^2
\]

The FOC gives Country \( \alpha \)'s reaction curve,

\[
u_2^\alpha = \frac{A^\alpha - u_2^\beta - x_2}{2}
\]

Similarly, Country \( \beta \)'s reaction curve is

\[
u_2^\beta = \frac{A^\beta - u_2^\alpha - x_2}{2}
\]

The intersection of the two reaction curves determine the last period Nash equilibrium, which is the following pair of decision rules for the last period

\[
u_2^\alpha = \phi_2^\alpha(x_2) = \frac{2}{3} A^\alpha - \frac{1}{3} x_2 - \frac{1}{3} A^\beta
\]

\[
u_2^\beta = \phi_2^\beta(x_2) = \frac{2}{3} A^\beta - \frac{1}{3} x_2 - \frac{1}{3} A^\alpha
\]

The resulting feedback equilibrium stock \( x_3 \) is then:

\[
x_3 = x_2 + \phi_2^\alpha(x_2) + \phi_2^\beta(x_2) = \frac{1}{3} (x_2 + A^\alpha + A^\beta)
\]

The feedback-equilibrium payoff to Country \( \alpha \) for period 2 (including the terminal term \( S^\alpha = -\frac{1}{2} x_3^2 \)) is

\[
 W_2^\alpha(x_2) = A^\alpha \phi_2^\alpha(x_2) - \frac{1}{2} \left( \phi_2^\alpha(x_2) \right)^2 - \frac{1}{2} x_2^2 - \frac{1}{2} \left( x_2 + \phi_2^\alpha(x_2) + \phi_2^\beta(x_2) \right)^2
\]
Similarly for Country $\beta$,

$$W_2^{\beta}(x_2) = A^\beta \phi_2^{\beta}(x_2) - \frac{1}{2} \left( \phi_2^{\beta}(x_2) \right)^2 - \frac{1}{2} x_2^2 - \frac{1}{2} \left( x_2 + \phi_2^{\alpha}(x_2) + \phi_2^{\beta}(x_2) \right)^2$$

Working backward, at the beginning of period 1, Country $\alpha$ chooses $u_1^\alpha$ to maximize

$$B_1^\alpha + W_2^\alpha(x_2) \equiv A^\alpha u_1^\alpha - \frac{1}{2} (u_1^\alpha)^2 - \frac{1}{2} x_1^2 + W_2^\alpha(x_1 + u_1^\alpha + u_1^\beta)$$

Its FOC yields the period 1 reaction function

$$u_1^\alpha = \frac{-11x_1 - 11u_1^\beta - 2A^\beta + 7A^\alpha}{20}$$

Similarly, Country $\beta$’s period 1 reaction function is

$$u_1^\beta = \frac{-11x_1 - 11u_1^\alpha - 2A^\alpha + 7A^\beta}{20}$$

The intersection of these two reaction curves gives the equilibrium feedback decision rules in period 1:

$$\phi_1^\alpha(x_1) = \frac{18}{31} A^\alpha - \frac{11}{31} x_1 - \frac{13}{31} A^\beta$$

$$\phi_1^\beta(x_1) = \frac{18}{31} A^\beta - \frac{11}{31} x_1 - \frac{13}{31} A^\alpha$$

How does the Markov-perfect Nash equilibrium compare to the open-loop Nash equilibrium?

Let us consider a numerical example. Let $A^\alpha = A^\beta = 20$, and $x_1 = 2$. Then, if both countries use open-loop strategies, the equilibrium emissions are $u_1^\alpha = u_2^\alpha = 1.09$, $u_1^\beta = u_2^\beta = 5.27$, $x_2 = 4.18$ and $x_3 = 14.72$. The overall payoff of each player in the OLNE is $-6.41$.

In contrast, in the MPNE, the feedback decision rules, applied to $x_1 = 2$, yield period 1 emissions levels $u_1^\alpha = u_1^\beta = 2.51$, which result in $x_2 = 7.03$. The second period emissions are obtained from applying the decision rules $\phi_2^\alpha$ and $\phi_2^\beta$ to the stock level $x_2 = 7.03$, yielding emissions levels $u_2^\alpha = u_2^\beta = 4.32 < 5.27$. The resulting terminal pollution stock is $x_3 = 15.67 > 14.72$. The overall payoff of each country under the Markov-perfect Nash equilibrium is $-25.34$.

The above numerical example indicates that both players are better off if they both use open-loop strategies. Under the open-loop strategies, given $x_1$, each country commits to a time path of actions, regardless of what $x_2$ turns out to be.

However, in the absence of a mechanism to ensure that they honor their commitments, the open-loop equilibrium is difficult to sustain. Each player would believe that the other player will deviate from the committed action for period 2 if $x_2$ turns out to be different from the level implied by their committed
period 1 emissions. Suppose Country $\alpha$ believes this. Then it will deviate from the OLNE by increasing its period 1 emissions beyond its OLNE level, so as to increase $x_2$, anticipating that Country $\beta$ will then reduce its period 2 emissions, as dictated by the rule $\phi_2^\beta(x_2)$. Thus Country $\alpha$’s first period deviation from OLNE will increase its overall payoff because it manages to pollute more in period 1, while getting Country $\beta$ to pollute less in period 2. If Country $\beta$ suspects this opportunistic behavior, it will also deviate from first period OLNE emissions.

The above discussion indicates that while an OLNE might achieve higher overall payoff for both players, their mutual suspicion and opportunistic behavior will prevent an OLNE from being realized.

**Example 2: Markov-perfect Nash equilibrium in a discrete-time model with an infinite horizon**

One of the earliest studies of feedback equilibrium in dynamic exploitation of a common property resource is the fish-war model of Levhari and Mirman (1980), formulated in discrete time. See the Section on transboundary fisheries.

**Notes on multiplicity of equilibria**

In general, multiple equilibria can occur. For example, Kemp and Long (2009) show that there are two equilibria in a game of foreign aids involving two donor countries. In one equilibrium both countries give more than in the other equilibrium. Another example is the paper of Amegashie and Runkel (2008). Similarly, Akao (2009) obtains multiple equilibria. In a model of capital flights, Long and Sorger (2008) find an interior equilibrium, and corner equilibrium. Dockner, Long and Sorger (1996) find multiple equilibria in a transboundary pollution game in discrete time. When there are multiple equilibria, it is not clear which one is likely to prevail, though some sort of stability argument may help the equilibrium selection. Myerson (2009, p. 1111) points out that the existence of multiple equilibria is a “pervasive fact of life that needs to be appreciated and understood, not ignored by economists.” Multiplicity of equilibria occurs in nature as well. As Stephen Jay Gold (1993, p. 28) noted, “places with apparently identical vegetation, moisture, and temperature, might harbor shells of maximally different form.”

**Choice of equilibrium concepts**

It is worth noting that OLNE and MPNE can be thought of as based on two alternative assumptions about the ability of players to commit. In an OLNE, players commit to a whole time path of actions. In a MPNE, players cannot commit at all. Reinganum and Stokey (1985) argue that in some cases, players may be able to commit to actions in the near futures (e.g. by forward contracts), but not to actions in the distant future. They develop a simple model where a game begins at time 0 and ends at a fixed time $T$, and there are $k$ periods of equal lengths $\delta$, where $k\delta = T$. At the beginning of each period, agents can commit to a path of action during that period. The special case where $k = 1$ corresponds to the open-loop formulation, and OLNE is then the appropriate equilibrium concept. At the other extreme, where $\delta \to 0$, the appropriate equilibrium concept is MPNE.

The choice of equilibrium concepts is to some extent dependent on tractabil-
ity. The relative ease of finding an OLNE is one of its attractive features. Fudenberg and Levine (1988) find that both OLNE and MPNE have merits. For some examples of OLNE in the economics of natural resources, see Gaudet and Long (1994, 2003), and Benchekroun et al. (2009, 2010).

2.4 Stackelberg equilibrium in dynamic games

Let us turn to situations where players are not symmetrically placed. Consider a two-player game, and assume that player \(1\) can make a commitment on what strategy she will use before player \(2\) can choose his strategy. We call player \(1\) the Stackelberg leader, and player \(2\) the follower. The resulting equilibrium is called a Stackelberg equilibrium. In dynamic games, players make their moves in each period. There are several concepts of dynamic Stackelberg leadership which must be distinguished. There is an obvious distinction between open-loop Stackelberg equilibrium (where the leader is committed to a time-path of action) and feedback Stackelberg equilibrium (where the leader’s action is contingent on the state variable). Within the latter class, Basar and Olsder (1982), in particular, distinguish between the stagewise feedback Stackelberg equilibrium and the global feedback Stackelberg equilibrium.

2.4.1 Open-loop Stackelberg equilibrium

An open-loop Stackelberg leader knows that, for any given time path of her control variables which she announces at the start of the game, the follower will choose a best reply in order to maximize his payoff. The leader therefore can compute her payoff that would result from each of her feasible announced path, and choose the optimal one. Her best announced path, together with the best reply of the follower, constitute an open-loop Stackelberg equilibrium.

It turns out that, unlike open-loop Nash equilibria (which are time-consistent), open-loop Stackelberg equilibrium is generically not time-consistent: if at some time after the game has started the leader is relieved of her commitment to follow her pre-announced path, she will typically find it optimal to deviate from it. (There are exceptions, see Dockner et al., 2000.) The intuition behind this time-inconsistency is as follows. If you promise to pay someone a stream of rewards on the condition that he carries out some investment, then once the investment has been sunk, you will have no incentives to keep your promise (given the implicit assumption that there is no loss of reputation, or no cost arising from a loss of reputation). In contrast, in an open-loop Nash equilibrium, because of the simultaneous choice of time paths, no player is trying to influence the action of any other players.

The time-inconsistency of open-loop Stackelberg equilibrium does not mean that this equilibrium concept is useless. It is a useful equilibrium concept in situations where the leader can credibly pre-commit, for example by signing a contract that is perfectly enforceable. For example, university teachers in Canada are often required to announce in advance, at the beginning of the term, what topics will be covered in the next thirteen weeks, in what order,
what articles students must read, the date a which the midterm exam will be held, what is the percentage of final grade for each assignment, and so on. Deviations will be punished (for example, in some cases promotion can be denied for serious deviations). Thus, a Canadian university teacher is often an open-loop Stackelberg leader, while the students are the followers.

An early model of open-loop Stackelberg leadership in natural resource economics is Kemp and Long (1980). They assume there are three players: player 1 is a major oil-importing country (called Home), player 2 is the collection of perfectly competitive oil-extracting firm located in the second country (called Foreign), and player 3 are residents of the rest of the world (called ROW). All consumers have the same demand curve, and their demand for oil is zero if the price of oil is greater than or equal to \( \alpha \) (we call \( \alpha \) the choke price). Assume zero extraction cost. Then Hotelling’s Rule tells us that producer’s price must increase over time at a rate equal to the rate of interest. Assume player 1 acts as an open-loop Stackelberg leader. It announces at the beginning of the game a time path of per-unit tariff rate. Kemp and Long (198) show that its optimal tariff rate increases over time at the rate of interest (so that, in Home, consumer’s price rises at the rate of interest, i.e., Hotelling’s Rule also applies to consumers). It follows that in the Stackelberg equilibrium, at some finite time \( T \) the tax-inclusive price for consumers in Home reaches \( \alpha \), while in ROW the price is still below \( \alpha \). From \( T \) onwards, oil is consumed only in ROW, until the resource stocks are exhausted. The open-loop Stackelberg equilibrium found in Kemp and Long (1980) is well defined. However it is based on the assumption that player 1 is able to commit to the sequence of tariff rates that it initially announces. This assumption does not seem plausible: at time \( T \), player 1 will have an incentive to reduce the tariff to allow imports into Home. In this sense, the initially announced tariff path is “time-inconsistent.”

### 2.4.2 Stagewise feedback Stackelberg equilibrium

Since open-loop Stackelberg equilibria are generically time-inconsistent, it is natural to formulate a concept of feedback Stackelberg equilibrium that ensures time-consistent. Consider a two-player dynamic game where, even though both players are intertemporal optimizers, the leader only “leads” in each period. Both players know this, and they solve their problems backwards, beginning with the last period, \( T - 1 \). The equilibrium strategies and equilibrium payoffs for period \( T - 1 \) clearly depend on the state variable at the beginning of that period, say \( y_{T-1} \). Knowing this equilibrium, both players know what to do in period \( T - 2 \), and so on. This scenario has been referred to as stagewise Stackelberg equilibrium. \(^5\)

---

4The time-inconsistency issue arises in many other contexts, see e.g. Kydland and Prescott (1977), Cohen and Michel (1988). Karp (1984) explores the issue of time inconsistency of optimal tariff in open-loop Stackelberg leadership and proposes that a restriction be imposed on the Stackelberg leader’s behavior so that the strategy she announces at the beginning is time-consistent. Karp’s method has been applied to a number of interesting problems, see Batabyal (1996a, 1996b).

illustrate the concept of stagewise Stackelberg equilibrium. For examples, please see Section 4 (on transboundary fisheries).

**2.4.3 Global feedback Stackelberg equilibrium**

Under stagewise leadership, the leader perceives that in each period $t$, the follower’s move at $t$ is a reaction to her own move at $t$. This is not the same thing as saying that the follower’s decision rule is a reaction to the leader’s decision rule. A truly global feedback Stackelberg equilibrium would require that the leader knows how the follower’s choice of decision rule responds to each possible decision rule of the leader. This is a very difficult problem to formulate in general terms. Basar and Olsder (1982) noted that “such decision problems cannot be solved by utilizing standard techniques of optimal control theory [...] because the reaction set of the follower cannot, in general, be determined in closed form, for all possible strategies of the leader, and hence the optimization problem faced by the leader on this reaction set becomes quite an implausible one” (page 315). Attempts to formulate the problem of global feedback Stackelberg leadership include Dockner et al. (2000, Chapter 5), Shimomura and Xie (2008), and Long and Sorger (2010).

Suppose the leader uses a decision rule strategy $u^L = L(x)$ where $x$ is the state variable and $u^L$ is the leader’s control variable. Consider the HJB equation of the follower:

$$r^F V^F(x) = \max_{u^F} \left\{ B^F(x, u^F, \phi^L(x)) + \left( \frac{dV^F}{dx} \right) G(x, u^F, \phi(x)) \right\}$$

For each possible function $\phi^L(\cdot)$, in principle there is a reaction $\phi^F(\cdot, \phi^L(\cdot))$, such that

$$u^F(t) = \phi^F(x(t), \phi^L(\cdot))$$

As pointed out in Dockner et al. (2000, Chapter 5) the difficulty is how to find the leader’s best feedback strategy $\phi^L(x)$. An associated problem is whether the leader best feedback strategy, if it can be found, is time-consistent. (See Long and Sorger (2010) for a discussion of this time-consistency issue for the leader.)

Fortunately, for a small class of problems, a global feedback Stackelberg solution can be found. As an example, a government may want to design a tax rule that corrects for stock externalities. Benchekroun and Long (1998) show that in the case of a pollution stock contributed by emissions of oligopolists, there exists an optimal tax rule where the rate of tax per unit of emissions is made dependent on the state variable. This tax rule achieves the socially optimal outcome, which corresponds to the first-best control and command scenario.\[6\] Because the first best is achieved, there is no reason why the government would change the tax rule at a later date. In other words, the global feedback Stackelberg equilibrium

\[6\]Benchekroun and Long (2002a) show the multiplicity of optimal tax rules.
found by Benchekroun and Long (1998) is time-consistent. Similarly, in a model of international trade involving bilateral monopoly in an exhaustible resource market where the importing country acts as the global Stackelberg leader, Fujiwara and Long (2010) find a time-consistent optimal tariff that is conditioned on the state variable (the stock of resource that remains).

Long and Sorger (2010) formulate a model of stochastic differential game with transferable utility with the principal-agent interpretation. They fully characterize the equilibrium where the principal is the global feedback Stackelberg leader.

In what follows, we present an example of a global feedback Stackelberg equilibrium.

**Example: global feedback Stackelberg equilibrium: efficiency-inducing pollution tax**

This example draws from Benchekroun and Long (1998). A monopolist faces an inverse demand function $P = P(Q)$ where $P'(Q) < 0$. Its output is $Q$, which is privately costless to produce, but the social cost of production is positive: each unit of output generates one unit of emissions, which contributes to a pollution stock $X$. Assume that

$$\dot{X} = Q - \delta X$$

where $\delta$ is the rate of decay of pollution. The pollution stock $X(t)$ incurs damages to consumers. The aggregate damage at time $t$ is $D(X(t))$, where $D(X) > 0$ for all $X > 0$. Assume $D(X)$ is convex.

Let $\tau(t)$ be the tax rate at time $t$ per unit of output. In addition, the government gives the monopolist a flow of transfer payments $T(t)$. The profit of the monopolist is then

$$\pi(t) = P(Q(t))Q(t) - \tau(t)Q(t) + T(t)$$

The payoff of the monopolist is

$$V^M = \int_0^\infty e^{-rt} \pi(t) dt$$

The government perceives the social welfare at time $t$, denoted by $W(t)$, as a weighted sum of consumer surplus net of damages, $CS(Q(t)) - D(X(t))$, profit, $\pi(t)$, and tax revenue, $\tau(t)Q(t) - T(t)$.

$$W(t) \equiv \lambda_c [CS()] - D(X(t))] + \lambda_f [P(Q(t))Q(t) - \tau(t)Q(t) + T(t)] + \lambda_g [\tau(t)Q(t) - T(t)]$$

where consumer surplus is

$$CS(Q(t)) \equiv \int_0^{Q(t)} P'(Q')dQ' - P(Q(t))Q(t) \equiv u(Q(t)) - P(Q(t))Q(t)$$

The objective of the government is to

$$\max \int_0^\infty e^{-rt} W(t) dt$$
We will focus on the special case where \( c = f = g \). Then the government’s objective reduces to
\[
\max \int_0^\infty e^{-rt} [u(Q(t)) - D(X(t))] \, dt
\]
Consider the first best scenario where the government can directly control the output \( Q(t) \). Then its HJB equation is
\[
rJ(X) = \max_Q [u(Q) - D(X) + J'(X) (Q - \delta X)]
\]
Since \( D(X) \) is convex and \( u(Q) \) is strictly concave, this problem has a unique solution, and the optimal output can be expressed as a function of \( X \),
\[
Q^* = \phi(X)
\]
with the property that \( \phi(.) \) satisfies the differential equation
\[
\phi'(X) = \frac{(\rho + r)P(\phi(X)) - D'(X)}{(\phi(X) - \delta X) P'(\phi(X))}
\]
with the boundary condition
\[
\phi(X^*_\infty) - \delta X^*_\infty = 0
\]
where \( X^*_\infty \) is the steady-state stock, uniquely defined by the condition
\[
P(\delta X^*_\infty) = \frac{D'(X^*_\infty)}{\rho + \delta}
\]
Let us return to the case where the output is chosen by the monopolist. Can the government design a stationary Markovian tax-transfer rule that induces the monopolist to adopt a decision rule that is identical to the socially optimal policy function \( \phi(X) \)?

Let us restrict attention to a tax-transfer function of the form
\[
T(t) + \tau(t)Q = \mu(X) + \theta(X)Q
\]
Then the monopolist must solve the problem
\[
\max_Q \int_0^\infty e^{-rt} [P(Q)Q - \theta(X)Q + \mu(X)] \, dt
\]
The maximum principle yields the following Euler equation for the monopolist
\[
\frac{dQ^m}{dX^m} = \frac{\dot{Q}^m}{X^m} = \frac{[P'(Q^m)Q^m + \theta(X^m)] (r + \delta) - \delta X^m \theta'(X^m) + \mu'(X^m)}{[P'(Q^m)Q^m + 2P(Q^m)] (Q^m - \delta X^m)}
\]
The monopolist’s path approach a steady state \( X^m_\infty \) where
\[
(r + \delta) [P'(\delta X^m_\infty) \delta X^m_\infty + P(\delta X^m_\infty) - \theta(X^m_\infty)] = \delta X^m \theta'(X^m_\infty) - \mu'(X^m_\infty)
\]
To make the monopolist choose the same policy function \( \phi(X) \) as the social planner, we must make sure that \( \theta(.) \) and \( \mu(.) \) be such that

\[
\phi'(X) = \frac{[P'(\phi(X)))\phi + P(\phi(X)) - \theta(X)] (r + \delta) - \delta X \theta'(X) + \mu'(X)}{[P''(\phi(X)))\phi + 2P'(\phi(X))] (\phi(X) - \delta X)}
\]

(1)

\[
(r + \delta) [P'(\delta X_\infty)\delta X_\infty + P(\delta X_\infty) - \theta(\infty)] = \delta X_\infty \theta'(X_\infty) - \mu'(X_\infty)
\]

Let us set \( \mu(.) = 0 \) identically. Then we seek a function \( \theta(.) \) such that

\[
\phi'(X) = \frac{[P'(\phi(X)))\phi + P(\phi(X)) - \theta(X)] (r + \delta) - \delta X \theta'(X)}{[P''(\phi(X)))\phi + 2P'(\phi(X))] (\phi(X) - \delta X)}
\]

(2)

and

\[
(r + \delta) [P'(\delta X_\infty)\delta X_\infty + P(\delta X_\infty) - \theta(\infty)] = \delta X_\infty \theta'(X_\infty)
\]

(3)

Since we know the social optimum policy function \( \phi(.) \), condition (2) reduces

\[
\frac{(r + \delta)P(\phi(X)) - D'(X)}{P'(\phi(X))} = \frac{[P'(\phi(X)))\phi + P(\phi(X)) - \theta(X)] (r + \delta) - \delta X \theta'(X)}{[P''(\phi(X)))\phi + 2P'(\phi(X))]}
\]

(4)

Notice that if \( \theta^*(.) \) is a solution of eq. (4) subject to the boundary eq (3), then so does \( \theta^*(.) + \gamma(.) \) where \( \gamma(.) \) satisfies the condition

\[
\delta X \gamma'(X) + (r + \delta) \gamma(X) = 0
\]

i.e.

\[
\gamma(X) = K X^{-(r+\delta)/\delta}
\]

where \( K \) is a constant.

**Remark:** The value function for the monopolist will depend on \( K \). To see this, suppose that we pick an efficiency-inducing tax scheme \( \theta^*(.) \). Then the monopolist’s HJB equation is

\[
rV^*(X) = \max_Q \left[ P(Q)Q - \theta^*(X)Q + \frac{dV^*}{dX} (Q - \delta X) \right]
\]

(5)

which implies that

\[
P'(Q)Q + P(Q) - \theta^*(X) = - \frac{dV^*}{dX}
\]

(6)

Recall that \( Q \) is socially optimal, so \( Q = \phi(X) \). Thus

\[
P'(\phi(X))\phi(X) + P(\phi(X)) - \theta^*(X) = - \frac{dV^*}{dX}
\]

Now, define \( \theta^{**}(X) = K X^{-(r+\delta)/\delta} + \theta^*(X) \). Then \( \theta^{**}(.) \) is also an efficiency-inducing tax scheme. With this tax scheme, let \( V^{**}(.) \) be the monopolist’s HJB equation.
Then
\[ rV^*(X) = \max_Q \left[ P(Q)Q - (KX^{-(r+\delta)/\delta} + \theta^*(X))Q + \frac{dV^*}{dX}(Q - \delta X) \right] \] (7)

Hence
\[ P'(\phi(X))\phi(X) + P(\phi(X)) - KX^{-(r+\delta)/\delta} - \theta^*(X) = -\frac{dV^*}{dX} \] (8)

Thus, using (6) and (8), we get
\[ \frac{dV^*}{dX} + KX^{-(r+\delta)/\delta} = \frac{dV^*}{dX} \] (9)

hence
\[ V^*(X) = V(X^*) + \left( 1 - \frac{r + \delta}{\delta} \right)^{-1} KX^{1-r+\delta} + H \] where \( H \) is a constant (10)

It can be verified that \( H = 0 \).

**Theorem** (Benchekroun and Long, 2002): If \( \theta^*(X) \) is an efficiency-inducing per unit tax for a polluting monopolist, then so is \( \theta^{**}(X) \), where
\[ \theta^{**}(X) = KX^{-(r+\delta)/\delta} + \theta^*(X). \]

and the value function for the monopolist is augmented by the term \( (1 - \frac{r+\delta}{\delta})^{-1} KX^{1-r+\delta} \).

**Extension**: Efficiency-inducing tax rule for a polluting oligopoly

Consider a Cournot oligopoly with \( n \) identical firms producing a homogeneous good under constant marginal cost. Let \( \Phi(X) \) be the feedback control rule (policy function) of the social planner. Suppose the social planner wants to implement a symmetric equilibrium such that the output of each firm is \( \phi(X) \equiv \frac{1}{n} \Phi(X) \). Let \( \theta^*(X) \) be an efficiency-inducing per unit tax rule. Then the value function of each firm \( i \) satisfies the following HJB equation
\[ rV^*_i(X) = \max_{q_i} \left[ P((n-1)\phi(X) + q_i)q_i - \theta^*(X)q_i + \frac{dV^*_i}{dX}((n-1)\phi(X) + q_i - \delta X) \right] \]
such that
\[ P'(n\phi(X))\phi(X) + P(n\phi(X)) - \theta^*(X) = -\frac{dV^*_i}{dX} \]

It can be verified that \( \theta^{**}(X) \equiv KX^{-(r+\delta)/\delta} + \theta^*(X) \) is also an efficiency-inducing tax rule.

### 2.5 Differential Games with Special Structures

In general it can be quite difficult to solve a differential game. However, if the game satisfies some special structures, the task of finding a solution is much easier. Below are some special structures that facilitate the solution.

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2.5.1 Linear-quadratic differential games

In a linear-quadratic differential game, the period payoff of each player is quadratic in the control and state variables, while the transition equations are linear in these variables. To find a solution, it is usual to try a quadratic value function for each player, with the implication that each player’s Markovian strategy is linear in the state variables.

Note however that there may exist several Markov-perfect Nash equilibria with linear strategies. For a recent example, see Amegashie and Runkel (2008). Even with the linear-quadratic structure, there may exist also Markov-perfect equilibria where agents use non-linear strategies (in which, of course, the value functions are not quadratic). Examples include Tsutsui and Mino (1990) and Dockner and Long (1993). Rowat (2007) provides a comprehensive treatment of this issue.

2.5.2 Quadratic differential games with transition equations that are homogeneous of degree one

This type of games is slightly more general than linear-quadratic differential games. Here is an example, drawn from Fujiwara and Matsueda (2009). There are two players, called player 1 and player 2. They share a public good (say a system of dykes which prevents flooding). The quality of the public good is denoted by \( G(t) \), and is a state variable. This variable can change over time. Let \( L_i(t) \) denote the effort that player \( i \) devotes to the improvement of \( G(t) \).

We assume that the rate of change of \( G(t) \) is given by the following transition equation:

\[
\frac{dG}{dt} = \left( L_1^\theta + L_2^\theta \right)^{1/\theta} - \delta G, \quad G(0) = G_0 > 0
\]

where \(-\infty < \theta \leq 1\) and \(\delta > 0\). Here \(\delta\) is the natural rate of depreciation. The restriction \( \theta \leq 1 \) ensures that the Hamiltonian of each player is concave in its control variable. The term \( \left[ L_1^\theta + L_2^\theta \right]^{1/\theta} \) can be thought of as a (constant elasticity of substitution) production function which combines individual efforts to create “quality improvement”.

The right-hand side of the transition equation is homogeneous of degree 1 in \((L_1, L_2, G)\).

The welfare of player \( i \) is

\[
W_i = \int_0^\infty e^{-rt} \left( aG - \frac{b}{2}G^2 - \frac{c}{2}L_i^2 \right) dt \quad \text{where } r > 0, \text{ and } a, b, c > 0
\]

To find the MPNE, suppose that player 1 thinks that player 2 uses a linear feedback strategy of the form

\[
L_2 = \alpha_2 G + \beta_2
\]

Then player 1’s optimization problem is: choose \( x_1 \) to maximize

\[
W_1 = \int_0^\infty e^{-rt} \left( aG - \frac{b}{2}G^2 - \frac{c}{2}L_1^2 \right) dt
\]
subject to

\[ \dot{G} = \left[ L_1^\theta + (\alpha_2 G + \beta_2)^\theta \right]^{1/\theta} - \delta G \]

The HJB equation for player 1 is

\[ rV_1(G) = \max_{x_1} \left\{ aG - \frac{b}{2}G^2 - \frac{c}{2}L_1^2 + V_1'(G) \left[ \left[ L_1^\theta + (\alpha_2 G + \beta_2)^\theta \right]^{1/\theta} - \delta G \right] \right\} \]

We guess that the value function \( V_1(G) \) is quadratic in \( G \):

\[ V_1(G) = \frac{A}{2}G^2 + BG + D \]

Then

\[ V_1'(G) = AG + B \]

Assume symmetry so that both players use the same strategy, \( L = \alpha G + \beta \). Solving for \( A \):

\[ A = \frac{c(r + 2\delta) \pm \sqrt{c^2(r + 2\delta)^2 + 6 \left( \frac{2^{2-\theta}}{\theta} \right) bc}}{3 \left( \frac{2^{2-\theta}}{\theta} \right)} \]

We take the negative root to ensure the stability (i.e., effort levels fall as the stock rises).

Then

\[ \alpha = \frac{1}{c^{1-\theta}} A = \frac{2^{-1/\theta}c(r + 2\delta) - \left( \frac{2^{2-\theta}}{\theta} \right) \sqrt{c^2(r + 2\delta)^2 + 6 \left( \frac{2^{2-\theta}}{\theta} \right) bc}}{3c} < 0 \]

Similarly, we can solve for \( B \) and \( D \), and deduce that

\[ \beta = \frac{a}{c \left[ \frac{2^{2-\theta}}{\theta} (r + \delta) - 3\alpha \right]} > 0 \]

**Remark:** More generally, we can consider a transition equation of the form

\[ \dot{G} = F(L_1, L_2) - \delta G \]

where \( F(L_1, L_2) \) is homogeneous of degree 1 and concave in \( (L_1, L_2) \).

### 2.5.3 Linear-state differential games

Following Dockner et al (2000) we define a game as a linear-state game if the first order condition for maximizing the Hamiltonian is independent of the state variables, and the equations of motion of the co-state variables are independent of the state variables. An important property of linear-state games is that open-loop and feedback equilibria coincide (i.e., the feedback rules are degenerate).
Example 1

Let \( x \) be the state variable, and \( u_i \) is the control variable of player \( i \). Assume that

\[
\dot{x} = u_1 u_2 - \delta x
\]

and player \( i \) maximizes

\[
\int_0^\infty e^{-rt} \left[ \frac{1}{2} u_i^{1/2} + x \right] dt
\]

It can be shown that the Markov-perfect equilibrium strategies are independent of \( x \).

Remark: There are games that are not linear in the state variables, but by a simple transformation of variables, they can be converted to linear state games.

Example 2: Long and Sorger (2008) consider the transition equation

\[
\dot{x} = Ax - u_1 - u_2
\]

and the net benefit function

\[
B_i = F(u_i, x) - \kappa u_i
\]

where \( F(.) \) is homogeneous of degree 1 in \((u_i, x)\). Assume that \( F \) is strictly concave in \( u_i \). Define the new control variables \( c_i = \frac{u_i}{x_i} \). Let \( \psi_i \) be the co-state variable. Then the Hamiltonian of player \( i \) is

\[
H_i = x \left[ F(c_i, 1) - \kappa - \psi_i x \right] + \psi_i \right] [A - c_i - c_j]
\]

This becomes a linear-state differential game.

2.6 Timing Games

Timing games are not strictly differential games. The simplest timing games involve when to make an once-for-all investment. Examples of timing games include The games of technology adoptions that we survey in this sub-section are dynamic games without the usual transition equation. In the simplest technology adoption model, the relevant state of the system in the identity of the firms that have switched over to a new technology. See Reinganum (1981) and Fudenberg and Tirole (1985).

A recent development involves merging the timing game with the theory of real options. Huisman et al. (2003) provide a brief survey of this literature. The simplest model involves two potential firms, each capable of producing one unit of output after it enters the market. The decision is when to enter the market, if the market demand function has a parameter which is a random variable obeying a stochastic process.
Chapter 3 Capital Flights, Foreign Aid, War and Peace

How to improve the welfare of individuals in the developing world is one of the most difficult questions of this century. Many poor countries suffer from outward capital flights, bad use of foreign aid, and inter-state as well as intra-state conflicts. Do differential games offer some explanation of these phenomena?

3.1 Why does capital fly away from poor countries?

Economists presume that poor countries are in need of capital, and the rate of returns to capital in those countries is likely to be higher than that in developed economies. This has led to the prediction that capital will go from rich countries to poor countries. However, for many poor countries, the outflow of capital is greater than the inflow. The reason is that while the social rate of returns of investing in a poor country is indeed quite high, the private rate of returns (taking into account the possibility of returns being grabbed by rival fractions) is low. The differential games models by Tornell and his co-authors offer some insight into this problem.

Tornell and Velasco (1996) model the main economic activities in a poor economy as the exploitation of a common access renewable resource with a linear growth function. Extraction is equated to grabbing by powerful groups within the country. The model is capable of explaining why capital flows from poor countries to rich countries. The idea is developed further by Tornell and Lane (1999) who use the same model, but offer more intuition. They show that the Markov-perfect Nash equilibrium leads to slow growth and displays a “voracity effect”: an apparently shock, such as a terms-of-trade improvement, perversely generates a more-than-proportional increase in fiscal redistribution (to powerful groups) and reduces growth. This effect is particularly strong when the number of competing fractions is small.

3.1.1 Corruption, capital flight, and the voracity effect

Tornell and Velasco (1996) consider a country that has a stock of productive asset, $k$. There are $n$ rivalrous fractions that exploit this resource. Let $u_i(t)$ be the rate of extraction by group $i$ at time $t$. The objective function of group $i$ is to maximize its infinite-horizon payoff function

$$\int_0^\infty \left( \frac{\sigma}{\sigma - 1} \right) [u_i(t)]^{(\sigma - 1)/\sigma} \exp(-rt)dt$$

subject to

$$\dot{k}(t) = Ak(t) - u_i(t) - \sum_{j \neq i} c_j(t), \quad A \geq 0$$

where $u_i = 0$ if $k = 0$. Here $A$ is the natural growth rate of the resource, and $\sigma > 0$ is the intertemporal elasticity of substitution.
We look for a Markov-perfect equilibrium in linear strategies\(^7\). Suppose agent \(i\) believes that all other agents \(j \neq i\) use a linear feedback strategy \(u_j = \alpha_j k\). Let
\[
\beta = \sum_{j \neq i} \alpha_j
\]
Without loss of generality, write \(u_i(t) = \alpha_i(t) k(t)\). Then agent \(i\)’s optimization problem is to find a time path \(\alpha_i(t) \geq 0\) that maximizes
\[
\int_0^\infty \left( \frac{\sigma}{\sigma-1} \right) k^{(\sigma-1)/\sigma} [\alpha_i(t)]^{(\sigma-1)/\sigma} \exp(-rt) dt
\]
subject to
\[
\dot{k}(t) = \left[ A - \beta - \alpha_i(t) \right] k(t)
\]
The Hamilton-Jacobi-Bellman equation is
\[
r V_i(k) = \max_{\alpha_i} \left\{ \left( \frac{\sigma}{\sigma-1} \right) k^{(\sigma-1)/\sigma} + V_i'(k) \left( A - \beta - \alpha_i \right) k \right\}
\]
Let us conjecture that the value function takes the form
\[
V_i(k) = \frac{z \sigma}{\sigma - 1} k^{(\sigma-1)/\sigma}
\]
where \(z\) is to be determined. Maximization of the right-hand side of the Hamiltonian gives
\[
\alpha_i = z^{-\sigma}
\]
substituting this equation into the HBJ equation, one gets
\[
\alpha_i = z^{-\sigma} = \sigma r + (1 - \sigma)(A - \beta)
\]
This equation shows that if \(\sigma < 1\) (the bounded utility case), then \(\beta\) is a strategic substitute for \(\alpha_i\): if a fraction’s rivals increase their intensity of exploitation, it will reduce its own intensity (so as to conserve the resource). In contrast, if \(\sigma > 1\), then \(\beta\) is a strategic complement for \(\alpha_i\). In the case \(\sigma = 1\) (logarithmic utility), the dominant strategy is \(\alpha_i = r\) regardless of \(\beta\).

In the special case where \(n = 1\), the restriction \(\sigma r + (1 - \sigma)A > 0\) will ensure that \(\alpha_i > 0\). (Otherwise, a solution would not exist.)

Let us focus on the symmetric equilibrium, i.e., all agents choose the same intensity of extraction. The Nash equilibrium is
\[
\alpha^N = \frac{\sigma r + (1 - \sigma)A}{n - \sigma(n - 1)}
\]
where we assume \(n - \sigma(n - 1) > 0\). This means that \(\sigma\) cannot be too large. Clearly, if \(\sigma < 1\), an increase in the number of players will reduce \(\alpha^N\). If \(\sigma > 1\),
\footnote{For results on the class of common property resource games that admit linear Markov strategies, see Long and Shimomura (1998) and Gaudet and Lohoues (2007).}
an increase in $n$ will increase $\alpha^N$. The aggregate extraction is $n\alpha^N$, which increases with $n$.

Tornell and Velasco also consider the case where the players can hold private wealth (for example, in a foreign bank account) that yields a constant rate of return $r > 0$. Assume $A > r$. Each agent $i$ can then extract $d_i(t)$ from the common property asset, and deposit it in his private bank account. Assume that there are exogenous upper bound and lower bound on $d_i(t)$, such that $\theta_L k(t) \leq d_i(t) \leq \theta_H k(t)$. Consumption $c_i(t)$ is financed by withdrawing from the private account. Let the state variable $f_i(t)$ stands for the balance of this bank account. Then we have a system of $n + 1$ differential equations

$$\dot{k}(t) = A k(t) - d_i(t) - \sum_{j \neq i} d_j(t)$$

$$\dot{f}_i(t) = R f_i(t) + d_i(t) - c_i(t), \ i = 1, 2, ..., n.$$  

Tornell and Velasco find that there are three symmetric equilibria. First, there is an interior equilibrium where all players use the extraction strategy

$$d_i(t) = \beta^{int} k(t)$$

where $\theta_L < \beta^{int} < \theta_H$. In the pessimistic equilibrium, everyone extracts at the maximum rate: $d_i(t) = \theta_H k(t)$. In the optimistic equilibrium, extraction is at the lowest possible rate: $d_i(t) = \theta_L k(t)$.

To find an interior symmetric equilibrium, let us conjecture that the value function is of the form

$$V_i(k, f_i) = \frac{z^\sigma}{\sigma - 1} (k + f_i)^{(\sigma-1)/\sigma}$$

Solving the HJB equation, we find that

$$\beta^{int} = \frac{A - R}{n - 1} > 0$$

$$c_i(t) = z^{-\sigma} (k(t) + f_i(t))$$

This model shows that capital can flow from poor countries, where the rate of return is high ($A > R$) to rich countries where the rate of return is lower ($R$). The reason is that each fraction knows that while the social rate of return of holding asset in the form of $k$ is $A$, the private rate of return is only $A - (n - 1)\beta^{int}$, because it faces $(n - 1)$ rival groups who can appropriate part of the common return.

In the pessimistic equilibrium, each fraction extracts at the maximum rate, because each believes that all other players extract at the maximum rate. In this case, the value function has the following form:

$$V_i(k, f_i) = \frac{z^\sigma}{\sigma - 1} (q k + f_i)^{(\sigma-1)/\sigma}$$

---

8If $d_i(t) < 0$, this signifies that the individual transfers funds from his private account to the common asset (if this goes on for ever, $f_i(t)$ will eventually be negative, which the foreign bank would permit only if it can one day get hold of a fraction of $k$).
where \( q \) and \( z \) are to be determined. Solving the HJB equation, we get

\[
0 < q = \frac{\theta_H}{A - r + n\theta_H} < 1
\]

The extraction strategy is \( d_i(t) = \theta_H k(t) \), and the consumption strategy is

\[
c_i(t) = [\delta \sigma + r(1 - \sigma)] (qk(t) + f_i(t)) \equiv z^{-\sigma} (qk(t) + f_i(t))
\]

Tornell and Lane (1999) observe that the interior equilibrium of this model display what they call “the voracity effect”: an increase in \( A \) will lead to an increase in \( z \). This means if the economy experiences a technical progress (or perhaps a terms of trade improvement), all the players will extract more, leading to faster depletion of the common asset.

### 3.1.2 Extensions of the capital flight model

In the above models, the extraction from the common stock is costless and the authors claim that “including appropriation or adjustment costs would add nothing to the insights provided by the model” (Tornell and Lane, 1999). However Sorger (2005) and Long and Sorger (2006) argue that an explicit consideration of the costs of appropriation is important. A model that takes appropriation costs into account is not only more realistic (after all, activities such as money laundering and lobbying use up of real resources) but is also likely to yield new insight. Long and Sorger (2006) show that both an increase in the appropriation costs and, when appropriation costs vary across agents, an increase in the degree of heterogeneity of these costs reduce the growth rate of the common asset.

Consider a common property asset with constant rate of return \( R \). Long and Sorger (2008) interpret this asset as the aggregate capital stock in an economy with insecure property rights. There are \( n \) identical players with access to this asset. Each player represents a group of individuals who – because they are organized and because property rights are poorly defined or not fully enforced – can influence the allocation of the capital stock among the \( n \) groups. The extraction rate of agent \( i \) at time \( t \) from the common property asset is denoted by \( E_i(t) \), the asset stock itself at time \( t \) is denoted by \( z(t) \), and the initial value at time 0 is denoted by \( z_0 \). The aggregate capital stock evolves therefore according to the differential equation

\[
\dot{z}(t) = Rz(t) - \sum_{i=1}^{n} E_i(t) , \quad z(0) = z_0.
\]  

(12)

Resources that the agents extract from the aggregate capital stock can be either consumed or invested into private and secure assets. These private assets, denoted by \( y_i \), can be interpreted as safe bank accounts in foreign countries, where property rights are fully enforced. The rate of return on the private asset is constant and given by \( r \). Denoting the rate of consumption of agent \( i \) at time
by \( c_i(t) \), the investment into the private asset of agent \( i \) at time \( t \) is given by 
\[
E_i(t)(1 - \theta) - c_i(t)(1 + v),
\]
where \( \theta \in [0, 1) \) is the fraction of of \( x(t) \) that is lost in the process (we may call this the \textbf{money laundering cost}) and \( v > 0 \) is the proportional cost of withdrawing from the private asset (we may call this the \textbf{debit tax}). Then
\[
\dot{y}_i(t) = ry_i(t) + E_i(t)(1 - \theta) - c_i(t)(1 + v), \quad y_i(0) = y_{i0}
\]
Assume that the private asset holdings must be non-negative.

Let us define
\[
A_i(t) = y_i(t) + \gamma z(t),
\]
where \( \gamma \) is a non-negative parameter. We may interpret \( A_i(t) \) as the total wealth of agent \( i \) at time \( t \), where \( \gamma \) measures the weight given to public asset holdings relative to private asset holdings. The agents derive utility both from consumption and from wealth. Denote the instantaneous utility of agent \( i \) by 
\[
U(c_i(t), A_i(t)).
\]
The presence of \( A_i(t) \) as an argument of the utility function displays a wealth effect often ignored in the literature on growth under insecure property rights.

Extraction from the common property asset is also assumed to be costly. The marginal cost of appropriation from the common property asset measured in units of utility is assumed to be constant and will be denoted by \( \kappa \). Let the time-preference rate of all agents be \( \rho > 0 \). Agent \( i \) seeks to maximize
\[
\int_0^{+\infty} e^{-\rho t} [U(c_i(t), A_i(t)) - \kappa E_i(t)] \, dt
\]
subject to constraints (12)-(14).

Unlike Tornell and Lane (1999) and Tornell and Velasco (1992), Long and Sorger (2006) do not rely on iso-elastic utility functions. The functional form of the utility function can be arbitrary, as long as it possesses concavity and homogeneity of degree one in \((k, c)\). Long and Sorger find that an increase in the degree of heterogeneity of cost leads to slower growth, and under certain condition, a higher elasticity of substitution between wealth and consumption will lead to a higher intensity of extraction, and thus lower growth.

The above-mentioned models of capital flight have a common feature: the assumption that agents care only about their absolute consumption levels (and possibly absolute wealth levels). That assumption ignores the fact that individuals do care about relative consumption (or relative income). An individual is happier the more her consumption (or income) level exceeds the average of her reference group. The following question then arises: if agents who exploit a common property renewable resource (either in the literal sense, or in the figurative sense) care about their \textit{relative} performance, would the social welfare and the growth rate of the public asset be more adversely affected compared to the case where they care only about their absolute performance? This question is dealt with in Long and Wang (2008).

Think of a lake that is effectively a common property to a number of municipalities, or provinces. Suppose the leader of each municipality is rewarded
according to some relative performance criterion, such as relative employment levels or relative local (municipal) GDP growth rates. Would these leaders have stronger incentive to allow local businesses to pollute the lake? Long and Wang (2008) explore the effect of the concern for relative consumption (or relative income) on the tragedy of the commons, both in the sense of common access natural resources, and in the sense of rent-seeking fiscal appropriations. They take as starting point the model of Tornell and Lane (1996, 1999), where powerful groups grab revenue from a common-access resource, and assume that agents gain utility from both absolute consumption and relative consumption. Long and Wang (2008) also consider the case where agents differ with respect to some characteristics. They introduce two sources of heterogeneity: agents may differ with respect to the degree of status-consciousness as well as with respect to appropriation cost. They show that social welfare decreases if agents become more heterogeneous in terms of status-seeking, but it increases if they become more heterogeneous in terms of appropriation costs.

3.2 Foreign Aid in the Presence of Corruption

There is a large literature on static models of foreign aid. The classic transfer problem (see, for example, Samuelson, 1952) may be thought of as an early analysis of some aspects of foreign aid. Kemp (1984) notes that international transfers, when conceived as voluntary contributions to a public good, are independent of small changes in the distribution of wealth among donor countries. Kemp and Shimomura (2002) remark that the theory of voluntary and unrequited international transfer rests on two incompatible assumptions: (i) each country is indifferent to the wellbeing of other countries, and (ii) voluntary unrequited international transfers do take place. They therefore propose a more satisfactory model that would allow for the possibility that the wellbeing of each country is influenced by the wellbeing of other countries, and characterize the optimal foreign aid from the point of view of the donor. Kemp, Long, and Shimomura (1992) formulate a model of dynamic foreign aid with capital accumulation, in the spirit of an infinite horizon principal agent model (but without moral hazard).

Kemp and Long (2009) offer two models of foreign aid where there are several donor countries that do not coordinate their aid strategies. In the first model, donor countries continually feel the warm glow from the act of giving. The feedback Nash equilibrium aid strategies turn out to be linear strategy \( A_i = \alpha X \), where \( X(t) \) is the capital stock of the recipient country and \( A_i(t) \) is the flow of aid from donor country \( i \). In the case of two donor countries, under certain restrictions on parameter values, there exist two symmetric equilibria, one with low aid, and one with high aid. At both equilibria, each country uses a linear Markov-perfect strategy. Interestingly, a lower degree of corruption in the recipient country is associated with a higher low-aid equilibrium, and with a lower high-aid equilibrium. If donor countries are status-conscious, it can be shown that the higher is the extent of status-consciousness of the donors, the greater is the sum of aids at the low-aid equilibrium, and the smaller is the sum...
of aids at the high-aid equilibrium.

In their second dynamic game model of foreign aid, Kemp and Long (2009) assume that there are two donor countries, and one recipient country. The only state variable is the “level of development” of the recipient country, denoted by $X(t)$. Assume that when $X$ reaches some level $\bar{X}$, the recipient country’s economy can take off and achieve sustained growth without help from abroad. The donor countries want the recipient to achieve the target $\bar{X}$, and the game ends when this target is reached. Let $A_i(t)$ be the flow of aid from donor country $i$. Assume that there is an upper bound on aid, so that $0 \leq A_i(t) \leq \bar{A}$.

Starting from any $X < \bar{X}$, the level of development $X(t)$ evolves according to the following dynamic law

$$\dot{X} = \beta_1(X)A_1 + \beta_2(X)A_2 + \omega(X)A_1A_2 - \delta(X)$$

for $0 \leq X < \bar{X}$

where $\beta_i(X) > 0$ is the effectiveness of country $i$’s aid. The term $\omega(X) \geq 0$ represents the interactive effect of the two flows of aids. The function $\delta(X)$ represents the depreciation of $X$. All the functions $\beta_i(\cdot), \omega(\cdot)$ and $\delta(\cdot)$ are differentiable, and bounded, for all $X \in [0, \bar{X}]$. The payoff of donor $i$ is assumed to be

$$J_i = K_i(X(T)) - \int_0^T c_iA_i(t)dt$$

where $c_i > 0$ is the cost per unit of aid, and $K_i(\cdot)$ is the psychological reward at the end of the program. The main results are that the following pair of strategies constitutes a Markov perfect Nash equilibrium,

$$\phi_i(X) = \frac{\delta(X)}{\beta_i(X)}, \quad i = 1, 2$$

and the value function of donor country $i$ is

$$V_i(X) = K_i(\bar{X}) - \int_X^{\bar{X}} \frac{\beta_j(x)c_i}{\beta_1(x)\beta_2(x) + \omega(x)\delta(x)}dx, \quad i = 1, 2$$

Three points are worth noting:

(i) The strategy (16) is, in general, non-linear in $X$. For example, consider the following specification of $\bar{X}$, $\delta(\cdot)$ and $\beta_i(\cdot)$:

$$\bar{X} = 1$$

$$\delta(X) = 1 - \exp\left[X - \bar{X}\right] \text{ for all } X \in [0, \bar{X}]$$

$$\beta_i(X) = \alpha_i + \delta(X)$$

where $\alpha_i > 0$. Then it can be shown that $\phi'_i(X) < 0$ and $\phi''_i(X) < 0$, that is, as the recipient country’s level of development grows, aid from each donor falls at a faster and faster rate. This reflects the fact that at the end of the horizon, the shadow price of the state variable has a value of zero.
(ii) The equilibrium growth rate of the stock $X$ is

$$
\dot{X} = \beta_1(X)A_1 \dot{r} + \beta_2(X)A_2 \dot{r} + \omega(X)A_1 A_2 - \delta(X)
$$

(18)

until $\tilde{X}$ is attained.

(iii) An increase in corruption can be represented as a downward shift in the function $\beta$. It follows from (16) and (18) that the higher is the level of corruption, the greater is the flow of aid, and the greater is the growth rate of the stock $X$ (unless $\omega(X) = 0$, in which the growth rate of the stock is independent of the level of corruption.)

3.3 War and Peace


Countries often fight because of the hope to gain territories. But they also fight because they seek revenge. Frank (1988) argues that emotions play an important role in conflicts. Jeager and Paserman (2007) provide an empirical analysis of fatalities in the Palestinian-Isreali conflict. They find that the data failed to reveal any revenge motive. Amegashie and Runkel (2008) offer a differential game model which show that “revenge matters.” What they discover is the “the Paradox of Revenge”: If both parties to a war derive pleasure from revenge, and both know this, they will exercise more restraint in their attacks. Therefore the the steady state will show lower intensity of conflicts than in a model without a revenge motive. Their model is as follows.

Let $D_i(t)$ is the summary measure of the past damages that country $i$ inflicts on country $j$. Let $u_i(t)$ be the intensity of fighting of country $i$’s army. Assume that

$$
\dot{D}_i(t) = u_i(t) - \delta D_i(t)
$$

where $\delta > 0$ is the rate of decay of the memories of one’s suffering.

At any period $t$, the probability that $i$ wins the battle in that period is

$$
\pi_i = \frac{1}{2} + \eta [u_i(t) - u_j(t)]
$$

where it is assumed that the absolute value of $\eta [u_i(t) - u_j(t)]$ is bounded by $\frac{1}{2}$.

If $i$ wins, its perceived prize is $\omega + R(D_j(t))$ where $D_j(t)$ is a summary measure of its past suffering. The effort cost of fighting at intensity $u_i$ is $C^i(u_i)$. Country $i$ seeks to maximize

$$
\int_0^\infty e^{-rt} [\pi_i [\omega + R(D_j(t))] - C^i(u_i(t))] dt
$$
Assuming that $R(D_i^j) = \alpha D_i^j$ where $\alpha > 0$, and

$$C^i(u_i) = \frac{1}{2} c_i u_i^2 \text{ where } c_i > 0$$

The game becomes a linear quadratic game. If country $i$ thinks that country $j$ uses a feedback strategy $u_j = g^j(D_j^i, D_i^j)$, then its HJB equation is

$$rV_i(D_j^i, D_i^j) = \max_{u_i} \left\{ \pi^i \left[ \omega + \alpha D_j^i(t) \right] - C^i(u_i(t)) + \frac{\partial V_i}{\partial D_j^i}(u_i - \delta D_j^i) + \frac{\partial V_i}{\partial D_i^j} \left[ g^i(.) - \delta D_i^j \right] \right\}$$

where $\pi^i = \frac{1}{2} + \eta [u_i - g^i(.)]$. Amegashi and Runkel (2008) solve the model numerically, using linear strategies and quadratic value functions. They find that there are three Markov-perfect Nash equilibria leading to three different stationary state. Two of these equilibria involve steady-state fighting intensity that is lower than that would prevail if there were no revenge motive.

So much for war. What about peace? It can be argued if the spirit of cooperation is a necessary condition for peace. In an article titled “The Build Up of Cooperative Behavior among Non-cooperative Agents,” Benchekroun and Long (2008) show that if each agent believes that all agents condition their behavior on the collective history of “good deeds” performed toward each other, then it is in each agent’s interest to build up a history of good deeds. Thus a Pareto efficient outcome can be achieved in the long run even if each agent is ultimately selfish.

In Benchekroun and Long (2008), the history of cooperation is a stock of intangible public good. This is in contrast to the standard dynamic public good literature where the stock of public good is tangible, see e.g. Itaya and Shimomura (2001), Kessing (2007), Fujiwara and Matsueda (2009).

## 4 Transboundary Pollution Games

Transboundary pollution has become a subject of increasing concern. In particular, the risk of substantial global warming damages has led policy makers to consider coordinated action of mitigation against GHG emissions. There are on the other hand strong incentives for individual countries to free ride on the mitigation efforts of other countries. To understand the dynamic trade-offs and motives of players, it is useful to formulate some dynamic game models of transboundary pollution. The first-generation models of dynamic games of transboundary pollution include Kaitala et al. (1991, 1992a, 1992b), Long (1992), Ploeg and de Zeeuw (1992), Hoel (1992, 1993), and Dockner and Long (1993). Asymmetric transboundary pollution (such as an upstream country that inflicts environmental damages on a downstream country) was considered by Jorgensen and Zaccour (2001).
4.1 Am simple transboundary pollution game

Let us consider the following game proposed by Long (1992) and Ploeg and De Zeew (1992). There are two countries. Country i’s output at date \( t \) is denoted by \( y_i(t) \). Assume all output is consumed. Emissions are proportional to output, \( E_i(t) = y_i(t) \) where the factor of proportionality is normalized at unity. (In what follows, we use \( y_i \) and \( E_i \) interchangeably.) The stock of pollution, common to both countries, is \( S(t) \). The rate of accumulation of the stock is equal to the sum of emissions minus the natural decay:

\[
\dot{S}(t) = E_1(t) + E_2(t) - \delta S(t) \tag{19}
\]

where \( \delta > 0 \) is the decay rate. The pollution damage suffered by country \( i \) at time \( t \) is

\[
D_i(S(t)) = \frac{c_i}{2} (S(t))^2
\]

where \( c_i > 0 \) is the damage parameter. The utility of consumption is

\[
U_i(y_i(t)) = A_i y_i(t) - \frac{1}{2} (y_i(t))^2 \tag{19}
\]

where \( A_i \) is a positive constant. The net utility, denoted by \( B_i(t) \) is defined as the utility of consumption minus the damage cost.

\[
B_i(t) = A_i y_i(t) - \frac{1}{2} (y_i(t))^2 - \frac{c_i}{2} (S(t))^2
\]

The government of country \( i \) perceives that the country’s social welfare is

\[
W_i = \int_0^\infty e^{-\rho t} B_i(t) dt
\]

where \( \rho > 0 \) is the rate of discount. Its objective is to maximize the country’s social welfare subject to the transition equation (19). In doing so, it must know if the government of the other country uses an open-loop or a feedback emission strategy. These two cases are examined separately below.

4.1.1 Open-loop Nash equilibrium in the infinite horizon model

If country \( i \) believes that country \( j \) uses an open-loop emission strategy, \( E_j(t) = \phi_j^{OL}(t) \), its optimization problem becomes

\[
\max_{E_i(.)} \int_0^\infty e^{-\rho t} \left[ A_i E_i(t) - \frac{1}{2} (E_i(t))^2 - \frac{c_i}{2} (S(t))^2 \right] dt \tag{20}
\]

subject to

\[
\dot{S}(t) = E_i(t) + \phi_j^{OL}(t) - \delta S(t), \quad S(0) = S_0 \tag{21}
\]

This is a simple optimal control problem, where \( \phi_j^{OL}(t) \) is taken as an exogenously given time path.

To find an open-loop Nash equilibrium, we must find a pair of functions \( \left( \phi_1^{OL}, \phi_2^{OL} \right) \) such that \( \phi_1^{OL}(t) = E_1^*(t) \) and \( \phi_2^{OL}(t) = E_2^*(t) \) where \( (E_1^*(t), E_2^*(t), S^*(t)) \) satisfy the three differential equations

\[
\dot{E}_1(t) = c_1 S(t) + (\rho + \delta)(E_1(t) - A_1) \tag{22}
\]
\[ \dot{E}_2(t) = c_2 S(t) + (\rho + \delta)(E_2(t) - A_2) \]  
\[ \dot{S}(t) = E_1(t) + E_2(t) - \delta S(t), \quad S(0) = S_0 \]  
(23)  
(24)

The transversality conditions are

\[ \lim_{t \to \infty} e^{-\rho t} (E_1(t) - A_1) = 0 \]  
\[ \lim_{t \to \infty} e^{-\rho t} (E_2(t) - A_2) = 0 \]  
(25)  
(26)

Let us consider the special case where the cost and preference parameters of the two countries are identical, i.e., \( A_1 = A_2 = A \) and \( c_1 = c_2 = c \). In this case, it is reasonable to assume that the two countries will adopt the same open-loop strategies, i.e., \( E_1^*(t) = E_2^*(t) = E^*(t) \). Then the system reduces to two differential equations

\[ \dot{E}(t) = c S(t) + (\rho + \delta)(E(t) - A) \]  
\[ \dot{S}^*(t) = 2E(t) - \delta S(t), \quad S(0) = S_0 \]  
(27)  
(28)

with the transversality condition

\[ \lim_{t \to \infty} e^{-\rho t} (E(t) - A) = 0 \]  
(29)

The pair of differential equations (27)-(28) admits a unique steady state pair \((\hat{S}, \hat{E})\) where

\[ \hat{S} = \frac{2A(\delta + \rho)}{2c + \delta(\delta + \rho)} \]  
(30)

\[ \hat{E} = \frac{A\delta(\delta + \rho)}{2c + \delta(\delta + \rho)} = \frac{\delta \hat{S}}{2} \]  
(31)

Hence, along the open-loop equilibrium play, there is a linear relationship between \( E^* \) and \( S^* \):

\[ E^* = \frac{1}{2} \left[ (\lambda_1 + \delta)S^* - \lambda \hat{S} \right] \]  
(32)

This equation is sometimes called the “feedback representation of the open-loop Nash equilibrium”. It should not be interpreted as a feedback strategy.

The case of asymmetric countries are considered in Long (1992) and reviewed in Long (2010).

To find a Markov-perfect Nash equilibrium, suppose country \( i \) believes that country \( j \) uses a feedback emission strategy, \( E_j(t) = \phi_j^{FB}(S(t)) \). Then the optimal control problem for country \( i \) is

\[ \max_{E_i(\cdot)} \int_0^\infty e^{-\rho t} \left[ A_i E_i(t) - \frac{1}{2} (E_i(t))^2 - \frac{c_i}{2} (S(t))^2 \right] dt \]  
(33)

subject to

\[ \dot{S}(t) = E_i(t) + \phi_j^{FB}(S(t)) - \delta S(t), \quad S(0) = S_0 \]  
(34)
Country $i$ realizes that if it influences $S$, it will indirectly influence the emission rate of country $j$. This adds a strategic consideration which is not present in the open-loop case.

The HJB equation for country $i$ is

$$
\rho V_i(S) = \max_{E_i} \left[ A E_i - \frac{1}{2} E_i^2 - \frac{c}{2} S^2 + V'_i(S) (E_i + E_j(S) - \delta S) \right] \tag{35}
$$

where $E_j(S)$ is country $j$’s feedback strategy. We impose the transversality condition

$$
\lim_{t \to \infty} e^{-\rho t} V(S(t)) = 0 \tag{36}
$$

Let us focus on a symmetric solution, where $V'_i(S) = V'_j(S) = V'(S)$ and $E_i(S) = E_j(S) = E(S)$. Then, substituting $E$ by $A + V'(S)$ into the HJB equation, we get

$$
\rho V(S) = \frac{1}{2} \left[ A^2 + 4AV' + 3(V')^2 \right] - \delta SV' - \frac{c}{2} S^2 \tag{37}
$$

This is a first order differential equation, which, together with condition (36), helps determine a MPNE.

Let us conjecture that the value function is quadratic

$$
V(S) = -\frac{\alpha S^2}{2} - \beta S - \mu \tag{38}
$$

Then we get a linear feedback strategy

$$
E(S) = A - \beta - \alpha S \tag{39}
$$

We expect that $\alpha > 0$, i.e., a higher stock will lead countries to reduce emissions, and $\beta > 0$, i.e., if $S = 0$, the marginal effect on welfare of an exogenous increase in $S$ is negative.

We obtain the following values for $\alpha$, $\beta$ and $\mu$:

$$
\alpha = \frac{1}{3} \left[ - \left( \delta + \frac{\rho}{2} \right) + \sqrt{\left( \delta + \frac{\rho}{2} \right)^2 + 3c} \right] \equiv \alpha_m \tag{40}
$$

(We will show below that for convergence to a steady state, we must choose the positive root of $\alpha$).

$$
\beta = \frac{2A \alpha}{\delta + \rho + 3\alpha} \equiv \beta_m
$$

$$
\mu = \frac{(A - \beta)}{2\rho} (3\alpha - \delta - \rho) \equiv \mu_m
$$

The linear feedback strategy is

$$
E = \frac{A(\delta + \rho + \alpha)}{\delta + \rho + 2\alpha} - \alpha S
$$
From these, we obtain

\[ S = \frac{2A(\delta + \rho + \alpha)}{\delta + \rho + 2\alpha} - (2\alpha + \delta)S \]  

(41)

For \( S \) to converge to a steady state, it is necessary that \( 2\alpha + \delta > 0 \). It can be verified that this condition is satisfied if and only if we take the positive root for \( \alpha \). The steady-state pollution stock under the MPNE using linear feedback strategy is

\[ \hat{S}^{M} = \frac{2A(\delta + \rho + \alpha)}{(\delta + \rho + 2\alpha)(2\alpha + \delta)} \]  

(42)

Comparing the OLNE steady state pollution stock \( \hat{S} \) (see equation (30)) with the MPNE steady state \( \hat{S}^{M} \), we see that the former is smaller than the latter.

Long (1992) also considers the case of open-loop Stackelberg equilibrium under fairly general assumptions: countries may have different discount rates and different damage cost parameters. An interesting result is that the leader’s optimal policy may result in a limit cycle. A dynamic game between asymmetric countries is also considered by Martin et al. (1993).

Dockner and Long (1993) find that in addition to a unique Markov-perfect Nash equilibrium where countries use linear feedback strategies, there exists a continuum of Markov-perfect Nash equilibria where the countries use non-linear feedback strategies.

Uncertainty has been introduced into the model of Dockner and Long (1993) by Bruno Nkuiya (2011). He assumes that over any small interval of time \( dt \) there is a positive probability \( \beta dt \) that the damage parameter increases from \( c \) to \( c + m \). He shows that in a Markov-perfect Nash equilibrium, both countries will choose a lower emission rate at any stock level, compared with the case where \( \beta = 0 \). The emission rate will jump up as soon as \( c \) increases to \( c + m \). Thus the treat of an upward jump in the damage parameter induces countries to be more cautious before the jump.

4.2 Other models of transboundary pollution with linear dynamics

Hoel (1992, 1993) sets up a finite horizon model in discrete time, and shows how equilibria differ when countries play open-loop and feedback strategies. A tax is introduced to achieve efficiency. See also Haurie and Zaccour (1995).

List and Mason (2001) consider an asymmetric version of the model of Dockner and Long (1993). They assume there are two regions with different parameters for the regional damage function and production function. They show that the outcome of a game between the two regions may be superior, in terms of social welfare, to that under an imperfect central planner who forces the two regions to have the same emission rate. List and Mason (1999) obtain similar results numerically for a problem where the central planner lacks information about the synergistic effects of two types of pollutants.
Dockner, Long and Sorger (1996) investigate a discrete-time, infinite horizon model of global warming with a catastrophic threshold. The pollution stock, denoted by $p_t$, is bounded above by $\overline{p}$, interpreted as the critical level beyond which the system will collapse. The damage cost functions become infinite for $p > \overline{p}$. The players are two countries with utility functions that are linear in output (which equals emissions). They prove that if the countries co-operate, the pollution stock will converge, in finite time, to a steady-state level $p^* < \overline{p}$. In contrast, if they do not cooperate, there are two types of Markov-perfect Nash equilibria. The first type of equilibrium is called Most Rapid Approach Path equilibrium (MRAP). The second type of equilibrium, called MTOI, for “Make-the-opponent-indifferent” equilibrium, displays the properties that each country is indifferent among all of its feasible choices. A numerical example is constructed where the MTOI equilibrium generates a chaotic path of pollution.\(^9\)

Yanase (2005) generalizes the model of Dockner, Long and Sorger (1996) by allowing the possibility that countries suffer from both flow and stock externalities, where the “flow externalities” may include oligopolistic market interactions. He also discusses trigger strategies that ensure cooperative outcomes.

Dutta and Radner (2006, 2009) consider strategic actions in a global warming model set in discrete time and infinite horizon. In their 2006 paper, they consider a world consisting of $I$ countries with exogenous population growth. The damage cost is linear in the stock of pollution. As is well known, when the stock enters the utility function linearly, the control variable (here the rate of emissions), when optimally chosen, will be independent of the stock level. They therefore find that in a Markov-perfect equilibrium, a country’s emission is independent of the greenhouse gas (GHG) stock, and dependent only on its own population. Assuming that a country’s emission of GHG is related to its energy input by a factor of proportion $f_i$, they show that if $f_i$ is initially large, a decrease in it will lead to an increase in the equilibrium emission rate.

### 4.3 Transboundary pollution with non-linear dynamics

We now turn to the case where the evolution of the pollution stock is non-linear. Examples of non-linear dynamics include lakes, grasslands, and coral reef systems.\(^{10}\) This is often referred to as the shallow lake problem. Suppose two countries share a shallow lake. Farms in each countries dump pollutants into the lake. These affect the ecosystem in a complex way, resulting in the possibility of multiple steady states and hysteresis effects. Let $u_i(t)$ be the action taken by agent $i$. In terms of the lake ecosystems, these actions would be phosphorus loadings into the lake as the result of fertilizer run-offs. The pollution damages caused by the eutrophication of the lake.

The evolution of the stock of pollution, $x(t)$, in the natural system is assumed to follow the following law:\(^3\)

\(^9\)For similar examples, see Dutta and Sundaram (1993b) and Dockner and Nishimura (1999).

\[
\dot{x} = \sum u_i(t) - \delta x + f(x), \; \delta > 0.
\]
where \( f(x) \) is an increasing, non-linear function. Typically, it is assumed that \( f(x) \) has a convex section, followed by a concave section. A common used specification is

\[
f(x) = \frac{x^2}{1 + x^2}
\]

The pollution stock \( x \) reduces the flow of useful services provided by the ecosystem. This is represented by a damage function \( D(x) \) which is convex and increasing. The applications of fertilizer yields a flow of benefits (e.g. agricultural output) represented by \( B(u_i) \) for agent \( i \). The differential game among the agent arises because each agent maximizes its private payoff, taking the strategy of other agents as given:

\[
\max \int_0^\infty e^{-rt} [B(u_i) - D(x)] \, dt
\]

As a benchmark, let us consider the outcome under the cooperative scenario. It is as if there is a central planner that seeks

\[
\max \int_0^\infty e^{-rt} \left[-nD(x) + \sum B(u_i) \right] \, dt
\]

Assume that \( B(u_i) = \ln u_i \) and \( D(x) = cx^2 \). Then the Euler equation is for the planner is

\[
\dot{u} = -[r + b - f'(x)] u + 2cxu^2
\]

As shown by Brock and Starrett (2003), the above pair of differential equations yield, in the \((u, x)\) space, an odd number of steady states, of which the middle one is unstable, while the other two are local stable in the saddle-point sense. Any steady state with \( x_{\infty} > 0 \) is a solution of the equation

\[
 bx^* + f(x^*) = \frac{r + b - f'(x^*)}{2cx^*}
\]

ehence

\[
\sum u_{i\infty}^* = \frac{r + b - f'(x^*)}{2cx^*}
\]

On the other hand, if the players do not cooperate, they will achieve an inferior payoff. In particular, in the open-loop Nash equilibrium, at a steady state, the following equation holds

\[
 bx_{\infty}^{OL} + f(x_{\infty}^{OL}) = \frac{[r + b - f'(x_{\infty}^{OL})]}{2cx_{\infty}^{OL}} n
\]

ehence, at that steady state,

\[
\sum u_{i\infty}^{OL} = \frac{[r + b - f'(x_{\infty}^{OL})]}{2cx_{\infty}^{OL}} n
\]
Mäler et al. (2003) shows that the steady-state concentration of phosphorous is higher in the OLNE, i.e. \( x_{OL} > x^* \). This is the usual manifestation of the tragedy of the commons. They show that a time-dependent tax on loading can achieve the social optimum. They also consider a time-independent tax per unit of loading, set at

\[
\tau = \frac{n - 1}{\sum u_i^*}
\]

In a numerical example, they find that if \( n \leq 7 \), then \( \tau \) would guide the agents to achieve \( x_{\infty}^* \). However, if \( n > 7 \), under the fixed tax rate \( \tau \) the agents can achieve \( x_{\infty}^* \) only under some set of initial conditions, while under other sets, they would attain a non-optimal steady state.

Markov-perfect Nash equilibria of this game has been found numerically by Kossioris et al. (2008) using the HJB equation. The MPNE steady state can be quite far from the socially optimal \( x_{\infty}^* \). In a subsequent paper, Kossioris et al. (2011) determine a Markovian tax rate \( \tau(x) \) per unit of loading that would guide agents to achieve the cooperative optimum. The necessary conditions that characterize a symmetric equilibrium Markovian strategy are derived in the same way as in Benckehroun and Long (1998). However, instead of finding the optimal tax rule, Kossioris et al. (2011) do not insist on achieving an optimal path of loading. They are contented with achieving an optimal steady state. Even so, a given tax function \( \tau(x) \) may give rise to a multiplicity of symmetric Markov-perfect Nash equilibrium strategies, leading to different steady states, even for the same initial state \( x_0 \). One only hopes that the agents will coordinate their choice of strategies.

### 4.4 Empirical Models of Transboundary Pollution Games

Empirical models are built with the purpose of quantifying the likely impacts of proposed policies on specific countries for a specific period of time. Real world data are used to calibrate parameters of demand and cost functions. The equilibrium paths of the model are then solved by numerical methods. Early empiricals include Kaitala et al. (1991, 1992a, 1992b, 1995). These papers deal with transboundary pollution and acid rain games involving a number of countries, including Finland and Russia.

Bernard, Haurie, Vielle, and Viguier (2008) set up a dynamic game model of trading in pollution permits where the two dominant players are Russia and China. Russia has a stock of permits to sell, and China could earn a lot of permits through the “Clean Development Mechanism,” one of the three flexibility mechanisms established by the Kyoto Protocol. Russia has a stock of permits \( x_1(t) \) that

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11 A static model of duopoly in the sales of permits is formulated by Loschel and Zhang (2002), where the two players are Eastern Europe and Russia.

12 Under the Kyoto Protocol, Annex B countries are consist of OECD countries plus Russia and Central and Eastern European countries.

13 The other two mechanisms are Joint Implementation (JI) and International Emission Trading (IET). Through JI, an Annex B country may implement a project that reduces
are banked at time $t$. The stock evolves according to the following law of motion

$$x_1(t+1) = x_1(t) - u_1(t) + q_1(t) + h(t)$$

where $u_1(t)$ is the amount of permits it sells in the world market, $q_1(t)$ is its level of emissions abatement, which earns new tradeable permits, and $h(t)$ is its exogenous flow of permits (called hot air).\textsuperscript{14} The stock $x_2(t)$ of the second dominant player (China) follows a similar law of motion, except that China does not have a hot air flow $h(t)$.

The authors assume that the price of a permit at time $t$, $p(t)$, is obtained by equating the demand by the rest of the world, $D_R(p(t))$, to the combined supply of permits by Russia and China:

$$D_R(p(t)) = u_1(t) + u_2(t)$$

(43)

where the function $D_R(.)$ is derived from the competitive equilibrium conditions for the Annex B countries (less Russia). This formulation assumes that Annex B countries do not take expected future prices of permits into account when they form their current demand. Inverting the market clearing condition (43) one gets

$$p(t) = D_R^{-1}(u_1(t) + u_2(t)) = P(u_1(t) + u_2(t))$$

(44)

Both China and Russia know that they can influence the price of permits. So they behave like duopolists in an exhaustible resource market (here, the resource stocks are $x_1(t)$ and $x_2(t)$). The authors assume that neither China nor Russia cares about the damage cost of global warming. Player $i$ maximizes

$$T \sum_{t=0}^{T-1} \beta_i^t [P(u_1(t) + u_2(t))u_i(t) - c_i(q_i(t))] + \beta_i^T \pi_i x_i(T)$$

where $\beta_i$ is the discount factor, $c_i(.)$ is the abatement cost function, and $\pi_i$ is the scrap value per unit of the final stock of permits.\textsuperscript{15} This is a difference game (De Zeeuw and van der Ploeg, 1991). The search for an equilibrium is formulated as a non-linear complementary problem, for which efficient algorithms have been developed (Ferris and Munson, 2000).

The authors numerically compute the open-loop Nash equilibrium. (It would be much more difficult to compute feedback Nash equilibria.)\textsuperscript{16} To do the calibration, they use simulation results of a computable general equilibrium model, namely GEMINI-E3 (Bernard and Vielle, 1998, 2000, 2003), and a partial emissions in another Annex B country. Through IET, an Annex B country may sell emission credits to another Annex B country.

\textsuperscript{14}After the fall of the Soviet Union, Russia experienced a sharp decline in output, which gives rise to emission credits. These are called hot air because they are available at no cost.

\textsuperscript{15}A stochastic variant of the model is Haurie and Viguier (2003), where random shocks are independent of the controls.

\textsuperscript{16}Yang (2003) proposes a method of solving for closed loop strategies by decomposing them into a sequence of open-loop ones, and implements the algorithm on the RICE model of Nordhaus and Yang (1996), which was formulated as an open-loop game.
equilibrium model of the world energy system, namely POLES (Criqui, 1996). GEMINI-E3 provides the data to estimate demand for pollution permits by Annex B countries, and marginal abatement costs for Russia. POLES is used to estimate the marginal abatement costs for China.

The Nash equilibrium is then compared with the outcome of an alternative scenario where only Russia behaves strategically. The authors find that duopolistic competition between Russia and China in their permit sales is likely to lower the permit price significantly.

### 4.5 Carbon Taxes under Bilateral Monopoly

The models mentioned above do not take into account the fact that the additional CO$_2$ accumulation in the atmosphere comes from the use of fossil fuels, which are extracted from non-renewable stocks (oil fields, coal mines). The dynamics of GHG accumulation is therefore closely linked to the dynamic of resource extraction. To the extent that the world’s flow of oil supply is under the control of a cartel, policies to curb GHG emissions (such as carbon taxes) will induce responses by the cartel. This should be taken into account in policy design, as Sinn (2008) recently emphasizes. It is therefore imperative that we develop models to gain insights into the dynamic interactions between tax policies and the pricing or extraction strategies of resource cartels. In what follows, we survey some theoretical attempts in that direction.

The simplest models consider only one state variable, namely accumulated extraction, and equates it with stock pollution. This may be justified on the grounds that the rate of decay of atmospheric CO$_2$ concentration is very low.

#### 4.5.1 Nash equilibrium under costless extraction and non-decaying pollution

Wirl (1994) considers a dynamic game between the government of a fossil-fuel importing country and a monopolist seller of fossil fuels extracted costlessly from a stock of resource $R$. It is assumed that the consumption of fossil fuels takes place only in the importing country, and the amount consumed, $q$, generates two adverse environmental effects: a flow pollution, with damages equal to $\frac{1}{2}q^2$, and a stock pollution, with damages equal to $\frac{1}{2}\delta Z^2$ where $Z$ is the stock of pollution, which is assumed to be the same as accumulated consumption, with $Z(0) = 0$ and $\dot{Z}(t) = q(t)$. Assume the stock of pollution does not decay. The monopolist exporter sets the producer price at each point of time, while the importing country sets a tax rate $\tau$ per unit. The demand function is $q = a - (p + \tau)$. The consumer’s surplus is then $\frac{1}{2}(a - p - \tau)^2$ for $p + \tau \leq a$.

The instantaneous welfare of the importing country at time $t$ is the sum of

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17 Bernard and Vielle (2002) and Bernard et al. (2003) model the dynamic monopoly of Russia in the permit market. Static models of maximizing Russia’s rent include Bohringer (2001), Buchner et al. (2002).
consumer surplus and the tax revenue, less the damage costs

$$W_M = \frac{1}{2}(a-p-\tau)^2 + \tau(a-p-\tau) - \frac{1}{2}\eta q^2 - \frac{1}{2}\delta Z^2$$

$$= \frac{1}{2}\left[(a-p)^2 - \tau^2\right] - \frac{1}{2}\eta(a-p-\tau)^2 - \frac{1}{2}\delta Z^2$$

The instantaneous welfare of the exporting country, $W_X$, is assumed to be equal to the seller's profit. Assuming that extraction cost is zero, we have

$$W_X = pq = p(a - (p + \tau))$$

Instantaneous global welfare is $W_G = W_M + W_X$. Then the steady state stock of pollution under the world social planner is $Z^\infty = \frac{a\tau}{\delta}$ provided that the initial resource stock is large enough, so that $\overline{R} \geq \frac{a\tau}{\delta}$.

In contrast, when the two countries play a non-cooperative game between themselves, the Nash equilibrium steady state pollution stock is higher. The oil-importing country believes that the exporting country uses a decision rule $p(t) = \phi(Z(t))$. Its instantaneous welfare at $t$ is

$$W_M(t, Z) = \frac{1}{2}\left[(a - \phi(Z))^2 - \tau^2\right] - \frac{1}{2}\eta(a - \phi(Z) - \tau)^2 - \frac{1}{2}\delta Z^2$$

Its objective is to maximize its welfare stream $J_M$ by choosing its tariff rates $\tau(t)$:

$$J_M = \max_{\tau(.)} \int_0^\infty e^{-rt}W_M dt$$

subject to $\dot{Z} = a - \phi(Z) - \tau$ and $Z(0) = 0, Z(t) \leq \overline{R}$. The solution of the importer’s HJB equation yields a decision rule $\tau = g(Z)$.

The exporter believes that the importer has a decision rule $\tau = \tilde{g}(Z)$. Its instantaneous welfare is $W_X = pq = p(a - p - \tilde{g}(Z))$. It chooses its producer price $p(t)$ to solve

$$J_X = \max_{p(.)} \int_0^\infty e^{-rt}p(a - p - \tilde{g}(Z)) dt$$

subject to $\dot{Z} = a - p - \tilde{g}(Z)$ and $Z(0) = 0, Z(t) \leq \overline{R}$.

It can be shown that the equilibrium feedback strategies are, for all $Z \leq a\tau/\delta$

$$\tau = g(Z) = a + \frac{2A^2 - 2\delta(2 + \eta)^2}{r(2 + \eta)^3} \left(\frac{a\tau}{\delta} - Z\right)$$

$$p = \phi(Z) = \frac{\delta(2 + \eta)^2 + 2bA^2}{r(2 + \eta)^3} \left(\frac{a\tau}{\delta} - Z\right)$$

Along the equilibrium path, the producer price falls monotonically to zero, and the tax $\tau$ rises steadily toward $a$. It can be shown that the time path of consumer
price in the Nash equilibrium is above the one that a world social planner would choose. This is because the seller aims at restricting output in the early phase of the program. Recall Hotelling’s famous dictum, “the monopolist is the conservationist’s best friend.”

Wirl (1994) argues that there exist other Nash equilibria where both players use non-linear strategies, and these equilibria lead to some steady state \( Z_1^* \). He also shows that both players would be better off by using linear strategies. One may argue that, in the context of this model, such equilibria are not subgame perfect, in the sense that if both players find themselves at \( Z_1^* \), they would want to move away from it, so that both would gain.

**4.5.2 Stagewise Stackelberg leadership in a pollution game with a fossil-fuel exporting country**

Tahvonen (1996, Section 3) considers a stage-wise Stackelberg game game using the set-up of Wirl (1994), without the flow externality, i.e. \( \eta = 0 \). The fossil-fuel exporting country is the leader. It turns out that the stagewise Stackelberg equilibrium for this model is identical to the Nash equilibrium found by Wirl (see Long, 2010, for a demonstration). This result suggests that it might be better to deal with the leadership issue by considering a model of global feedback Stackelberg games, which has been formulated in Long and Fujiwara (2009c) to the problem of optimal tariff in exhaustible resources.

**4.5.3 Bilateral monopoly under stock-dependent extraction cost extraction and non-decaying pollution**

Rubio and Escriche (2001) consider a variation of Wirl (1995) by assuming that extraction cost increases as the resource stock dwindles. As in Wirl (1995), let \( Z(t) \) denote accumulated extraction, and \( \dot{Z}(t) = q(t) \). Assume there is no flow pollution. The cost of extracting \( q(t) \) is \( cZ(t)q(t) \). They consider first the Nash equilibrium.

Rubio and Escriche (2001) then state that in the Markov-perfect Nash equilibrium, the tax \( \tau \) imposed by the fossil-fuel importing country is only a neutral Pigouvian Tax, in the sense that “it corrects only the inefficiency caused by the stock externality and leaves the cartel with its monopolistic profit.” However, in that model, consumers are not pure price-takers. They know the seller’s decision rule for wellhead price, \( p = \phi(Z) \), therefore when they choose \( q(t) \), they do take into account the fact that their demand \( q(t) \) will have an impact on the future level of \( Z \) and hence on the price they will pay in the future.

Rubio and Escriche (2001) show that (stagewise) Stackelberg equilibrium with the seller as the leader is identical to the Nash equilibrium. This result corresponds to the identity between the Nash equilibrium in Wirl (1994) and the (stagewise) Stackelberg equilibrium found by Tahvonen (1996, Section 3).

What happens if the importing country is the (stage-wise) Stackelberg leader? The solution reveals that while the long run stock \( Z_\infty \) under stagewise leadership of the importing country coincides with that obtained in the Nash equilibrium,
the equilibrium time paths of the stock under the two scenarios are quite different. The initial consumer price and tax are lower in the Nash equilibrium than under the stage-wise leadership of the importing country. The life-time payoff of the importing country is higher under its stage-wise leadership than under the Nash equilibrium, and the opposite result applies to the exporter's life-time profit.

Liski and Tahvonen (2004) study in more details the Nash equilibrium of the model of Rubio and Escribà (2001), where extraction cost rises as the stock of resource dwindles. Their focus is on the interpretation of the Nash equilibrium carbon tax in terms of a pure Pigouvian motive and a rent-shifting motive. Again, denoting the remaining stock of resource by $x = R - Z$, they find that the Nash equilibrium decision rule of the exporter, $p = p_N(x)$, can be an increasing function or a decreasing function. If there are no pollution damages (our $\delta = 0$), then $p_N(x)$ is a decreasing function, i.e. the wellhead price will be rising over time (as $x$ falls over time). If the damage costs are very high ($\delta$ is sufficiently great), then $p_N(x)$ is an increasing function, i.e., the wellhead price will be falling over time. The carbon tax, which consists of a Pigouvian component and a rent-shifting component, decreases (respectively, increases) over time if the damage parameter $\delta$ is small (respectively, large).

4.5.4 Models of carbon taxes with pollution decay

When the pollution stock has a positive decay rate, there is no longer an one-to-one relationship between the accumulated extraction and the pollution stock. One must then deal with a game involving two state variables.

The model by Wiril (1995) is an extension of his 1994 paper in two directions: the pollution stock has a constant rate of decay, and the extraction cost increases with cumulative extraction. The game now has two state variables: the stock cumulative energy production, $Z$, and a pollution stock, $S$. Analytical solution does not seem possible with two state variables, hence the author relies on numerical methods. In the special case where the decay rate is zero, the two state variables collapse to one, and analytical solution for a Nash equilibrium becomes possible. It is qualitatively very similar to that found in Wiril (1994). Tahvonen (1996) analyses a similar two-state variable game, but under the assumption that the the seller is the stage-wise Stackelberg leader.

4.6 International Environmental Agreements

While the motives for international environmental agreements can be studied in a static context, it is important to turn to a dynamic game formulation because stock pollution accumulates over time, and incentives may change when the stock level changes. Another reason for using a dynamic analysis is that because of political institutions, countries may delay joining an agreements, and the effects of delays are of course best studied in an explicitly dynamic framework.
4.6.1 IEA membership game

There is an extensive literature on international environmental agreements (IEAs), beginning with the static models of Carraro and Siniscalco (1993) and Barrett (1994). Building on the concept of self-enforcing IEAs introduced by Carraro and Siniscalco (1993) and Barrett (1994), Rubio and Casino (2005) consider the following two-stage game. In the first stage, countries decide whether to become member of an IEA. In the second stage, they choose their time paths of emissions over an infinite horizon. Signatory countries choose their emission paths cooperatively, while non-signatories act non-cooperatively. During the emission stage, countries cannot change their membership status.

Using numerical simulations, Rubio and Casino (2005) find that the only self-enforcing IEA is a two-country coalition. Germain et al. (2003), extending the static cooperative framework of Chandler and Tulkens (1995), show that it is possible to devise a dynamic transfer scheme to support the grand coalition. Petrosjan and Zaccour (2003) also use the cooperative approach to allocate cost burdens among members of the grand coalition.

Rubio and Ulph (2007) consider a discrete time, infinite horizon model, where IEA membership can vary over time. They assume that all countries are identical. After characterizing the fully non-cooperative equilibrium and the fully cooperative equilibrium, they consider the case where in each period, there is a two-stage game. In Stage 1, each decides whether to be a member of an IEA or not. In Stage 2, given the opening level of the pollution stock, signatories cooperatively choose their emission level, while each non-signatory chooses its own emission level non-cooperatively. An IEA is said to be stable (for a given period) if it is both internally stable and externally stable. At the beginning of each period, all countries have the same probability of being invited as a signatory. Even though the payoff in each period is quadratic, in the dynamic membership framework, the value function is not quadratic. Numerical solutions reveal three possible patterns of behavior. The most common pattern displays a negative relationship between the pollution stock and the number of signatories.

Nkuiya-Mbakop (2009) considers a variant of the model proposed by Rubio and Ulph (2007). He adopts a model in continuous time with an infinite horizon, and treats the length of the period of commitment (to stay with the IEA) as a parameter. Assuming there are \( N \) identical countries, he then studies the effect of varying the length \( \theta \) of the period of commitment on the equilibrium size of

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18 Useful surveys of this literature include Barrett (1997, 2003), Finus (2001), and Bencherroun and Long (2011).
19 Karp and Sacheti (1997) consider a two-period model of IEA with a stock pollutant and assess how the incentives to join an IEA are affected by the dynamics of the pollution stock, and the extent to which the pollution is global.
20 Many empirical models make the same assumption about commitment to membership, see e.g. Eyckman (2001).
21 They focus on the open-loop equilibrium.
22 Internal stability means that no signatory country would gain to switch to the non-signatory status; external stability means that no non-signatory would want to become a signatory.
the stable coalition and its relationship to the size of the pollution stock. Given
the pollution stock size $S$ at the beginning of each period, for any stable IEA of
size $n(S)$ in that period, all countries have an equal chance $n(S)/N$ of becoming
a signatory. Nkuiya-Mbakop (2009) shows that there are two critical values of
$\theta$, denoted by $\theta_L$ and $\theta_H > \theta_L$. If $\theta < \theta_L$, all countries want to be signatories.
If $\theta > \theta_H$, the only stable coalition is the two-country coalition. If $\theta \in [\theta_L, \theta_H]$, the greater is $\theta$, the smaller is the size of the stable coalition.

4.6.2 Effects of delays in signing an IEA

In a world consisting of two identical countries, there is an overwhelming in-
centive to cooperate, to maximize the sum of their welfare levels. Suppose,
however, that for some reasons the two countries cannot enter into an environ-
mental agreement immediately. Using the two-country model of Dockner and
Long (1993), Açikgöz and Benchekroun (2005) ask the following question. Sup-
pose the two countries know at time $t = 0$ that at a fixed time $T$ in the future
they will, with probability $\theta$, reach an agreement to cooperate from then on, but
if that opportunity is missed they won’t ever cooperate. What is the effect of
that possibility on their emission policies from time zero to time $T$? The answer
depends on whether they use Markov-perfect strategies or open-loop strategies
during the Phase I, i.e. during the time interval $[0, T]$.

Each country $i$ seeks to maximize its expected welfare

$$
\int_0^T e^{-\rho t} \left[ AE_i - \frac{E_i^2}{2} - \frac{c}{2} S^2 \right] dt + e^{-\rho T} \left[ \theta W^C + (1 - \theta) W^{NC} \right]
$$

where $W^C$ and $W^{NC}$ are the respective welfare levels of each country under
cooperation and under non-cooperation (the latter depending on whether the
open-loop Nash equilibrium or Markov-perfect Nash equilibrium will prevail
after $T$ given that an agreement is not reached at time $T$. Here $c > 0$ is a
parameter for pollution damages.

In Phase I, if countries use Markov-perfect strategies, their value function
will be of the form

$$
V_i(S, t) = -\frac{\alpha(t) S^2}{2} - \beta(t) S - \mu(t)
$$

where $\alpha(t)$, $\beta(t)$ and $\mu(t)$ are time-dependent parameters. From the necessary
conditions, $\alpha(t)$ and $\beta(t)$ must satisfy the following differential equations

$$
\dot{\alpha} = 3 \alpha^2 + \alpha(\rho + 2 \delta) - c \tag{45}
$$

$$
\dot{\beta} = \beta(\rho + 3 \alpha + \delta) - 2 A \alpha \tag{46}
$$

Solving to get

$$
\alpha(t) = \alpha_m + \frac{2 \gamma \exp(2 \gamma t)}{K_\alpha - 3 \exp(2 \gamma t)}
$$
where $K_n$ is a constant of integration and

$$
\gamma \equiv \sqrt{\left(\delta + \frac{\beta}{2}\right)^2 + 3c}
$$

Next, solve for $\beta(t)$, and denote the corresponding constant of integration by $K_\beta$. To determine the two constants of integration, one must use the following “smooth pasting” condition at time $T$, i.e.,

$$
\frac{\partial V_i(S,T)}{\partial S} = \theta \frac{dW^C}{dS} + (1 - \theta) \frac{dW^{NC}}{dS}
$$

This condition implies that $\alpha(T) = \theta \alpha_c + (1 - \theta) \alpha_m$ and $\beta(T) = \theta \beta_c + (1 - \theta) \beta_m$.

If countries use open-loop strategies in Phase I (and assuming that in Phase II, if the treaty is not signed, they will also use the open-loop strategies), a similar approach can be used to find the equilibrium of Phase I. For the open-loop case, the “smooth pasting” condition is the requirement that the shadow price at $T$ be equal to the weighted average of the two shadow prices under cooperation and under non-cooperation after $T$.

Açıkgoz and Benchekroun (2005) find that under the Phase I open-loop equilibrium, in anticipation of a possible international environmental agreement, countries will pollute more than they would under no anticipation. And in the case of Markov-perfect equilibrium in Phase I, countries will pollute so much more that the possible welfare gain from cooperation in Phase II is almost completely wiped out.

Breton, Sbraga, and Zaccour (2008) propose a different approach to games of IEAs. Using a discrete-time version of replicator dynamics, they assume that the group of countries that outperform others are copied by a fraction of new entrants to the game. The evolutionary process results in a stable IEA.

### 4.6.3 Coupling Constraint and Rosen Equilibrium

It is conceivable that an IEA might agree on a ceiling $\bar{E}(t)$ for the aggregate emission level $E(t)$ for each time $t$, but fail to allocate precise emission targets $E_i(t)$ for member countries. In such a case, how do member countries choose their emission levels? One possible answer is that each member country perceives that its maximum permissible emission depends on how much other member countries actually emit. Thus, for country $j$, it is required that

$$
E_j(t) \leq \bar{E}(t) - \sum_{i \neq j} E_i(t)
$$

Such a constraint is called a coupling constraint or a coupled constraint (see, e.g. Haurie, 1995, Haurie and Zaccour, 1995, Carlson and Haurie, 2000, Tidball and Zaccour, 2005 and 2009, Krawczyk, 2005, Bahn and Haurie 2008). Coupled constraints have important implications on the set of possible equilibria. In particular, there may exist a continuum of Nash equilibria. Rosen (1965) found that
when players must face coupling constraints, there are multiple Nash equilibria, even in a static game. To overcome this multiplicity, Rosen (1965) proposed the concept of normalised equilibrium, which he showed to be unique under certain regularity conditions. Krawczyk (2005) regrets that references to this concept are rare in the “main-stream economic” literature, and defends the usefulness of this equilibrium concept.

Consider a static model with two players, CAP and SMALL. Their choice variables are $Q$ and $q$ respectively. Suppose they faced the coupled constraint $q + Q \leq E$. Let $F(q, Q)$ and $f(q, Q)$ be their payoff functions. Rosen showed that in general there is a continuum of Nash equilibria, and that each Nash equilibrium corresponds to a “normalized equilibrium”, which he defined as follows.

Take an arbitrary weight vector $\rho \equiv (\rho_1, \rho_2) \geq (0, 0)$. Define the function $\theta(q, Q, q, Q, \rho) = \rho_1 f(q, Q) + \rho_2 F(q, Q)$. Consider the following problem: given $(q, Q, E, \rho)$, find $(q, Q)$ to maximize $\theta(q, Q, q, Q, \rho)$ subject to $q + Q \leq E$. Assume $f(q, Q)$ is strictly concave in $q$, and $F(q, Q)$ is strictly concave in $Q$. For fixed $(q, Q, E, \rho)$, this maximization problem yields a unique solution:

$$
\hat{q} = \hat{q}(\overline{q}, \overline{Q}, E, \rho) \text{ and } \hat{Q} = \hat{Q}(\overline{q}, \overline{Q}, E, \rho)
$$

A normalized equilibrium is defined as a fixed point of the mapping described by (47), i.e., it is a point $(\overline{q}^*, \overline{Q}^*)$ such that

$$
\hat{q}(\overline{q}^*, \overline{Q}^*, E, \rho) = \overline{q}^* \text{ and } \hat{Q}(\overline{q}^*, \overline{Q}^*, E, \rho) = \overline{Q}^*.
$$

Rosen (1965) showed that if each player’s payoff function is concave in his choice variable, then for any fixed vector $\rho \equiv (\rho_1, \rho_2) \geq (0, 0)$, there exists a normalized equilibrium. He also gave a sufficient condition for the uniqueness of normalized equilibrium given a fixed $\rho$. For this purpose, define the function

$$
\sigma(q, Q, \rho) \equiv \rho_1 f(q, Q) + \rho_2 F(q, Q)
$$

(notice that there is no upper-bar variables in this definition). The function $\sigma(q, Q, \rho)$ is said to be diagonally strictly concave in $(q, Q)$ if and only if for any $(q', Q')$ and $(q, Q)$ the following inequality holds

$$
\begin{bmatrix}
q - q' & Q - Q'
\end{bmatrix}
\begin{bmatrix}
\rho_1 f(q', Q') \\
\rho_2 F(q', Q')
\end{bmatrix}
+ \begin{bmatrix}
q - q' & Q' - Q
\end{bmatrix}
\begin{bmatrix}
\rho_1 f(q, Q) \\
\rho_2 F(q, Q)
\end{bmatrix}
> 0
$$

In his Theorem 4, Rosen (1965) showed that if $\sigma(q, Q, \rho)$ is diagonally strictly concave then for any fixed $\rho$, the normalized equilibrium is unique.

A troubling problem with the concept of normalised equilibrium is how the weights are determined. Notwithstanding this concern, there are dynamic game models with coupled constraints that use this equilibrium concept. See for example Haurie and Krawczyk (1997), Tidball and Zaccour (2005, 2009), Krawczyk (2005), and Bahn and Haurie (2008).
appropriate set of weights. For dynamic games, however, this property would
not hold unless the weights themselves are time-dependent (Tidball and Zaccour,
2009).

4.6.4 Games under the Kyoto Protocol

One of the flexible mechanisms encouraged by the Kyoto Protocol is the joint
implementation of environmental projects. Breton, Zaccour and Zahaf (2005) con-
sider a differential game model of foreign investments in environmental projects.
Under the assumption that the damage cost is linear in the pollution stock, they
formulate a continuous time, finite-horizon game between two asymmetric coun-
tries and compare three scenarios: Business-as-usual (BAU), where countries
are not committed to any environmental target; autarky, where each country
constrains its emissions to achieve a target pollution stock at the end of the
horizon; and joint implementations (JI) where countries can invest in environ-
mental projects abroad. The case of non-linear damage cost is taken up in
Breton, Martin-Herran, and Zaccour (2006) in a linear quadratic model. They
find that if joint implementation is allowed, both countries will invest in envi-
ronmental projects at home, and at least one will invest abroad. The identity
of the latter can change at most once during the planning horizon.

For other contributions to the analysis of game theoretic aspects of inter-
national cooperation, see Kaitala, Mäulet, and Tulkens (1995), Kaitala and

4.6.5 Some new directions for research on IEAs

Theoretical research on IEAs have mainly focuses on climate change mitiga-
tions (cutting back on emissions.) Another way of coping with global warming
has been relatively neglected: adaptation. Examples of adaptation include the
building of higher dykes, terracing in rural areas, improving the early warning
systems, etc. (See Parry et al. 2007). Unlike mitigation, which is a global public
good, adaptation is a local public good. Adaptation limits local damages from
climate changes and is a substitute for self-insurance (Auerswald et al, 2011).
There may be stronger incentives to undertake adaptation measures. Would
adaptation facilitate the formation of IEAs on mitigations? This question was
addressed in a static model by Marrouch and Ray Chaudhury (2011). They
found that when adaptation is taken into account, under certain model speci-
fications, emissions are no longer strategic substitutes: they can become strategic
complements. This fact can help increase the size of stable IEAs. A dynamic
game version of this model would include $n + 1$ state variables. The first $n$
state variables are the stocks of local public goods (such as dykes, levees, and other
preventive capital stocks) while the last state variable would be the global pub-
lic bad, namely the level of GHGs concentration (in excess of the pre-industrial
level).
5 Transboundary Fisheries

Transboundary fisheries are obvious instances of the tragedy of the commons (Gordon, 1954, Hardin, 1968). If countries have common access to a resource stock, they tend to over-exploit it. While some societies successfully develop institutions and norms of behavior that to some extent mitigate the tragedy of the commons (Ostrom, 1990), there are obvious cases of extreme over-exploitation. The Food and Agriculture Organisation reported that, in 2007, 80% of stocks are fished at or beyond their maximum sustainable yield (FAO, 2009). Grafton, Kompas, and Hilborn (2007) documented serious over-exploitation of several fish species. Recent empirical work by McWhinnie (2009) found that shared stocks are indeed more prone to overexploitation, confirming the theoretical prediction based on a dynamic game model of Clark and Munro (1975), that an increase in the number of players reduces the equilibrium stock level.

The “fishery model” has been interpreted more broadly to mean a model about almost any kind of renewable resource. This interpretation has been taken by many authors, e.g. Brander and Taylor (1997) and Copeland and Taylor (2009). Using a similar framework, Weitzman (1974) compares free access versus private ownership as alternative systems for managing common property, Chichilnisky (1994) considers a North-South trade model with common property resources. These four papers do not deal with situations involving dynamic games.

5.0.6 Feedback Nash equilibrium in a fish war

One of the earliest studies of feedback equilibrium in dynamic exploitation of a common property resource is the fish-war model of Levhari and Mirman (1980), formulated in discrete time. Below is a version of their model. Consider a stock of fish, denoted by $y_t$, shared by two countries, say country $i$ and country $j$. The harvest rates are $c_{it}$ and $c_{jt}$ and the transition equation is

$$y_{t+1} = (y_t - c_{it} - c_{jt})^\alpha, \quad 0 < \alpha < 1$$

where $c_{it} + c_{jt} \leq y_t$. The utility derived by country $k$ is $\ln(c_{kt})$.

Let us look at the cooperative scenario. The joint optimization problem is

$$\max_{t=0}^{\infty} \beta^t \left[ \ln(c_{it}) + \ln(c_{jt}) \right]$$

Define aggregate harvest by $h_t = 2c_t$. Then we seek

$$\max_{t=0}^{\infty} 2\beta^t \ln \left[ \frac{h_t}{2} \right] = \max_{t=0}^{\infty} 2\beta^t [\ln h_t - \ln 2]$$

Let $W(y)$ be the value function. Then the (discrete-time) Bellman equation for the joint maximization problem is

$$W(y_{t+1}) = \max \{2 \ln h_t - 2 \ln 2 + \beta W[(y_t - h_t)^\alpha]\}$$
Solving, we get the optimal aggregate harvest rule \( h_t = (1 - \beta \alpha) y_t \). This exploitation rule results in a time path of the stock \( y_t \) that converges to a unique steady state \( \hat{y} = (\beta \alpha)^{\alpha/(1-\alpha)} \).

Now suppose the two countries do not cooperate. Country \( i \) thinks that country \( j \) uses the harvesting strategy \( c^j_t = \phi^j(y_t) \), where \( \phi^j(0) = 0 \) and \( \phi^j(y) > 0 \). Country \( i \) takes \( \phi^j(\cdot) \) as given, and finds the path \( c^i_t \) to maximize \( \sum_{t=0}^{\infty} \beta^t \ln c^i_t \) subject to \( y_{t+1} = (y_t - \phi^j(y_t) - c^i_t)^\alpha \). The Bellman equation for country \( i \) is

\[
V^i(y_{t+1}) = \max_{c^i_t} \left\{ \ln c^i_t + \beta V^i \left[ (y_t - \phi^j(y_t) - c^i_t)^\alpha \right] \right\}
\]

Suppose country \( i \) thinks that country \( j \) uses the following linear harvest strategy \( \phi^j(y) = \gamma^j y \) where \( 0 < \gamma^j < 1 \). Let us conjecture that the value function of country \( i \) takes the form \( V^i(y) = A_i \ln y + B_i \). Then

\[
A_i \ln y + B_i = \max_{c^i_t} \left\{ \ln c^i_t + \beta A_i \alpha \ln (y - \gamma^j y - c^i_t) + \beta B_i \right\}
\]

We obtain the symmetric Markov-perfect equilibrium \( \phi(y) = \gamma y \) where

\[
\gamma = \frac{1 - \beta \alpha}{2 - \beta \alpha}
\]

The steady-state biomass is then

\[
\bar{y} = \left( \frac{\beta \alpha}{2 - \beta \alpha} \right)^{\alpha/(1-\alpha)}
\]

which is smaller than the steady-state stock under cooperation. This confirms that the tragedy of the commons can occur even if the harvesting function depends only on the effort and is independent of the stock.

The fish war model Levhari and Mirman (1980) has been generalised by Antoniadou et al. (2008) to the case where the utility function has a constant intertemporal rate of substitution \( \eta > 0 \), and the transition equation is

\[
s_{t+1} = \theta \left[ \alpha(y_t)^{\frac{\eta-1}{\eta}} + (1 - \alpha) k^{\frac{\eta-1}{\eta}} \right]^\frac{\eta}{\eta-1}
\]

where \( y_t = s_t - \sum c_{it} \) and \( k \) is a constant. Note the same parameter \( \eta \) appears in both functions. This specific coincidence permits the existence of a Markov perfect equilibrium with linear feedback strategies. When \( \theta \) is a random variable, Antoniadou et al. (2008) show that increases in risks do have an effect of the equilibrium feedback strategy (except in the case where \( \eta = 1 \)). If the support of the shock \( \theta \) is unbounded, conditions must be placed on the distribution of the shock to ensure existence of value functions\(^{23}\). Assuming that \( \ln \theta \) is normally distributed with mean \( \mu - \sigma^2/2 \) and variance \( \sigma^2 \) (so that \( E \theta = e^\mu \))

\(^{23}\)See Stachurski (2002) for sufficient conditions on the distribution of \( \theta \) in the case of a single player.
and \( \text{var}(\theta) = e^{2\mu(e^{\sigma^2} - 1)} \), Antoniadou et al. (2008) show that an increase in \( \sigma \) will amplify (respectively, mitigate) the tragedy of the commons if \( \eta < 1 \) (respectively, \( \eta > 1 \)). Another direction of generalization is to allow for several fish stocks having biological externalities, see Datta and Mirman (1999).

For additional contributions of dynamic games involving renewable resources, see Fischer and Mirman (1986), Benhabib and Radner (1992), Crabbé and Long (1993), Mason and Polasky (1994), Dockner and Sorger (1996), Benchekroun and Long (2002a), and Benchekroun (2003, 2008), Fujiwara, K. (2009), among others. The possibility of enforcing co-operation through the use of punishment activated by trigger strategies is discussed and analyzed in Dockner, Long and Sorger (1996). Problems of existence of a MPNE (with or without random disturbance) are discussed in Dutta and Sundaram (1993a,b) and Duffie et al. (1994).

5.1 Stagewise feedback Stackelberg equilibria in transboundary fisheries

An example of stagewise Stackelberg equilibrium in discrete time: transboundary fishery

This example is drawn from the fish-war model of Levhari and Mirman (1980). Assume that in each period, country 1 is the first mover: it announces how much it will catch in that period before country 2 makes its move. To find the stagewise Stackelberg equilibrium of this game, let us begin with the simplest case, where there is only one period to go, i.e., only one decision to make: how much to catch this period, given that next period (the terminal period) the game ends and player \( k \) \((k = 1, 2)\) receives a fixed share \( s^k \) of the final stock.

In the last period, Country 2 (the follower) takes the leader’s catch \( c^1 \) as given, and chooses \( c^2 \) to maximize

\[
\ln c^2 + \beta \ln s^2 \left[ y - c^1 - c^2 \right]^\alpha
\]

This yields country 2’s reaction function

\[
c^2 = \frac{y - c^1}{1 + \alpha \beta} = \phi_1^F(y, c^1)
\]

where the superscript \( F \) indicates that this is the follower's strategy, and the subscript 1 indicates that there is only one period to go.

Country 1, knowing this reaction function, chooses \( c^1 \) to maximize

\[
\ln c^1 + \beta \ln s^1 \left[ y - c^1 - \frac{y - c^1}{1 + \alpha \beta} \right]^\alpha
\]

This yields country 1’s decision rule \( g_1^L(\cdot) \):

\[
c^1 = \frac{y}{1 + \alpha \beta} = g_1^L(y)
\]
where the superscript $L$ indicates that this is the leader’s strategy, and the subscript 1 indicates that there is only one period to go. Substituting this decision rule into the follower’s reaction function, we obtain our prediction of the follower’s catch when there is only one period to go:

$$c^2 = \phi_1^{F}(y, g_1^{L}(y)) = \frac{\alpha \beta y}{(1 + \alpha \beta)^2} \equiv \psi_1^{F}(y)$$

Note that $\psi_1^{F}$ depends only on $y$ while the reaction function $\phi_1^{F}$ depends on both $c^1$ and $y$. We call $\psi_1^{F}$ the follower’s (one-period-to-go) equilibrium feedback strategy (as he predicts correctly the leader’s strategy). The terminal stock to be shared is

$$y - c^1 - c^2 = y - g_1^{L}(y) - \psi_1^{F}(y) = \frac{\alpha^2 \beta^2 y}{(1 + \alpha \beta)^2}$$

The follower’s payoff when there is only one period to go is thus given by the one-period-to-go value function $V_1^{2}(y)$:

$$V_1^{2}(y) = \ln \frac{\alpha \beta y}{(1 + \alpha \beta)^2} + \beta \ln \left[ s^2 \left( \frac{\alpha^2 \beta^2 y}{(1 + \alpha \beta)^2} \right)^\alpha \right] = \ln y + \alpha \beta \ln y + \text{constant}$$

$$= (1 + \alpha \beta) \ln y + \text{constant}$$

Armed with this information, we know that for the game where there are two periods to go, the follower will seek to maximize

$$\ln c^2 + \beta V_1^{2}(y - c^1 - c^2) = \ln c^2 + \beta (1 + \alpha \beta) \ln (y - c^1 - c^2)^\alpha + \text{constant}$$

i.e.,

$$\ln c^2 + (\alpha \beta)(1 + \alpha \beta) \ln (y - c^1 - c^2) + \text{constant}$$

To find the follower’s reaction function and the leader’s decision rule, a similar procedure applies. Repeating this process, we can see that in the limit (i.e. letting the time horizon tend to infinity), the stationary equilibrium feedback strategies of the leader and the follower are, respectively,

$$c^1 = (1 - \alpha \beta)y \quad \text{and} \quad c^2 = \alpha \beta (1 - \alpha \beta)y$$

This indicates that the leader has a higher payoff than the follower. The after-harvest stock is

$$y - c^1 - c^2 = \alpha^2 \beta^2 y$$

The stock will approach a steady-state level

$$y^S_\infty = \left[ \frac{\alpha^2 \beta^2}{\Gamma \alpha} \right]^{1/\alpha}$$

It is easy to see that the steady-state stock $y^S_\infty$ under the stagewise Stackelberg leadership is smaller than the steady-state stock $y^F_\infty$ under the feedback Nash equilibrium.
An example of stagewise Stackelberg equilibrium in continuous time: sequential exploitation of a migratory fish stock

Let us turn to a continuous-time model of stagewise Stackelberg leadership, based on Benchekroun and Long (2002b). The time horizon is infinite. Two countries have access to a migratory fish stock denoted by $S_t$. In each period, the fish travel along country 1’s coastline before reaching country 2. Country 1 is therefore a natural stagewise Stackelberg leader. If it chooses effort level $\omega_{1t}$, its harvest is $\omega_{1t}S_t$ and its profit is $(\omega_{1t}S_t)^\sigma$ where $0 < \sigma < 1$. Country 2, having observed $\omega_{1t}$, chooses its effort level $\omega_{2t}$. We assume that its profit function is $(\omega_{2t}S_t)^\sigma (1 - \omega_{1t})$, because country 1’s prior harvest makes it more costly for country 2 to catch fish. Assume that the transition equation is

$$\dot{S}_t = AS_t - \sum_{i=1}^{2} \omega_{it}S_t \equiv F(S_t, \omega_{1t}, \omega_{2t})$$

where we assume $A > 0$ and $r > \sigma A$. Suppose country 2 (the follower) believes that country 1’s strategy is $\omega_{1t} = g_1(S_t)$. Its HJB equation is

$$rV_2(S) = \max_{\omega_2} \{ (\omega_2 S)^\sigma (1 - g_1(S)) + V_2'(S)F(S, g_1(S), \omega_2) \}$$

Suppose that $g_1(S) = \omega_1$, a constant. Try the following value function for country 2: $V_2(S) = B_2 S^\sigma$. It can be verified that if $\omega_1 > 1$, then $B_2 = 0 = \omega_2$ and, if $0 \leq \omega_1 \leq 1$, then

$$\omega_2 = \frac{r - \sigma A + \sigma \omega_1}{1 - \sigma} \equiv \omega_2(\omega_1)$$

$$B_2 = \left( \frac{1 - \sigma}{r - \sigma A + \sigma \omega_1} \right)^{1-\sigma} (1 - \omega_1)$$

We now turn to country 1. Its HJB equation is

$$rV_1(S) = \max_{\omega_1} \{ (\omega_1 S)^\sigma + V_1'(S)F(S, \omega_1, \omega_2(\omega_1)) \}$$

Again, try the functional form $V_1(S) = B_1 S^\sigma$. Then, provided that $r - \sigma A < 1 - \sigma$, country 1’s HJB function is satisfied with

$$B_1 = (1 - \sigma) \left( \frac{1 - \sigma + \sigma^2}{r - \sigma A} \right)^{1-\sigma}$$

$$\omega_1 = \frac{r - \sigma A}{1 - \sigma} < 1$$

If $r - \sigma A < 1 - \sigma$ then this stagewise Stackelberg leader-follower game has the following solution. The leader’s strategy is

$$\omega_1 = g_1(S) = \frac{r - \sigma A}{1 - \sigma} \quad (48)$$

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and the follower’s strategy is

$$\omega_2 = \frac{r - \sigma A + \sigma \omega_1}{1 - \sigma} \quad (49)$$

which implies that in equilibrium

$$\omega_2 = \frac{\omega_1}{1 - \sigma} > \omega_1 \quad (50)$$

Notice that equation (49) may be interpreted as country 2’s reaction function, which displays strategic complementarity, namely, the reaction curve is upward-sloping. In equilibrium, the follower exercises greater effort than the leader.

It is interesting to note that had country 1 not known that $$\omega_2$$ depends on $$\omega_1$$, the result would have been a naive Nash-Cournot equilibrium, and country 1’s effort would be greater, at a level $$\bar{\omega}_1$$ defined by

$$\bar{\omega}_1 = \frac{r - \sigma A}{1 - 2\sigma} > \frac{r - \sigma A}{1 - \sigma}$$

(provided $$1 - 2\sigma > 0$$). We may therefore conclude that the stagewise Stackelberg leader exercises restraint because it knows the follower’s effort is an increasing function of that of the leader.

Benchekroun and Long (2002b) consider the possibility of an exogenous deviation from this equilibrium. For example, suppose the government of country 1, responding to pressures from conservationist groups, agrees to reduce its effort $$\omega_1$$ over a specified number of years. How would country 2 react? The authors find that the answer depends on the length of country 1’s period of committed reduction in harvesting. If the period of commitment is short, country 2’s optimal reaction is to increase its effort $$\omega_2$$. If it is long but finite, country 2 will initially reduce its harvesting effort, but will eventually increase it when the end of the commitment period is near. Finally, if the period of commitment is infinite, country will reduce its effort permanently.

5.2 Possibility of Pareto efficient Nash equilibria

In repeated games, if agents use trigger strategies to punish deviation from an efficient allocation, Pareto efficient outcomes can be achieved, provided that the discount rate is small enough. In dynamic games, where the state variables evolve over time, trigger strategies can also achieve a cooperative outcome (see e.g. Haurie and Pohjola, 1987, Benhabib and Radner, 1992). Trigger strategies require a complete memory of the history of moves, and therefore are not Markovian. Efficiency can also be achieved if agents can be induced to accept an “incentive equilibrium”, see e.g. Ehtamo and Hämäläinen (1989, 1993) and Jørgensen and Zaccour (2001b).24

The concept of “incentive equilibrium” is based on the assumption that player i can threaten player j that if j deviates from a cooperative path, i will punish j by behaving like a kind of follower in a static leader-follower game. It is not clear whether such punishment is fully rational.
Martin-Herrán and Rincón-Zapareto (2005) derive necessary conditions for a Markov-perfect Nash equilibrium to be Pareto efficient, and apply these conditions to a fishery game proposed by Clemhout and Wan (1985). In this game, despite common access, under certain parameter values, a Markov-perfect Nash equilibrium can be efficient if each player derives pleasure from other players’ consumption. This is not surprising: the positive externalities of altruism and the negative externalities of common access just happen to cancel each other out.

5.3 Some technical notes on feedback strategies in fishery problems

Feedback strategies in fishery problems are in general difficult to compute analytically. Analytical solutions turn out to be easy to find if each player’s problem can be transformed into an optimization problem that is linear in the (transformed) state variable in such a way that the first order condition with respect to the control variable is independent of the state variable. Recall the Levhari-Mirman model (1980). Suppose that each player chooses a linear decision rule

\[ c_i^t = \gamma_i^t s_t \]

where \( \gamma_i^t \in (0, 1) \). Then the transition equation is

\[ \ln s_{t+1} = \alpha \ln s_t + \alpha \ln(1 - \gamma_i^t - \gamma_j^t) \]

and using the transformation of variable \( z_t = \ln s_t \), we see that the transition equation is linear in \( z_t \). Similarly, the objective function of player \( i \) becomes

\[ \max_{z} \sum_{t=0}^{\infty} \beta^t [z_t + \ln \gamma_i^t] \]

The Bellman equation of player \( i \) then becomes linear in the new state variable \( z \).

A similar feature applies to models in continuous time: a pair of linear harvest rule, \( c_i^t = \gamma^t s \), \( i = 1, 2 \), constitutes a Markov-perfect Nash equilibrium strategies if by a suitable transformation of variables, the value function is linear in the transformed state variable. Let us illustrate. Suppose the stock of fish is \( x(t) \) and its net growth rate is

\[ \dot{x}(t) = Ax(t)^\theta - \delta x(t) - c_1(t) - c_2(t), \quad A > 0 \]

where \( c_i \) is the catch rate of country \( i \). Assume \( 0 < \theta < 1 \), \( \delta > 0 \) and \( A > 0 \). In this case the growth function \( Ax(t)^\theta - \delta x(t) \) is strictly concave. Since the derivative of this function, when evaluated at \( x = 0 \), is infinite, we can be sure that, starting with any \( x(0) > 0 \), the steady-state stock will be positive.

Assume the utility function is

\[ U(c_i) = \frac{c_i^{1-\beta}}{1-\beta} \]
Now make the very special assumption that $\beta = \theta$. Consider a transformation of the state variable, $X(t) \equiv x(t)^{1-\theta}$, and of the control variables by defining the variable $\omega_i(t)$ as the catch rate per unit of stock, i.e., $c_i(t) = \omega_i(t)x(t)$. Then

$$\dot{x}(t) = Ax(t)^\theta - (\delta + \omega_1(t) + \omega_2(t))x(t)$$

and (omitting the time argument),

$$\dot{X} = (1-\theta)x^{-\theta}\dot{x} = (1-\theta)x^{-\theta}[Ax^\theta - (\delta + \omega_1 + \omega_2)x]$$

Since $\beta = \theta$, the optimization problem is linear in the state variable.

The solution can then be obtained by appealing to the linear state property. Consider next the case of a U-shaped natural growth function that has a finite derivative at $x = 0$. Suppose

$$\dot{x}(t) = rx(t) - Bx^\eta(t) - c_1(t) - c_2(t)$$

where $r > 0$, $B > 0$, and $\eta > 1$. (The case $\eta = 2$ is the standard logistic growth function in the fishery literature). In this case, define the variable $X$ by $X(t) \equiv x(t)^{1-\eta}$. Notice that since $1 - \eta < 0$, a higher value of $X$ means a lower value of the true fish stock $x$. As $X \to \infty$, $x \to 0$.

To summarize, to have Markov-perfect equilibrium harvesting strategies of the form $c_i = \gamma x$ where $\gamma > 0$, there must exist a special relationship between the natural growth function and the utility function. This feature was stated in Clemhout and Wan (1985), and further generalized by Gaudet and Lohoues (2008).

Koulovatianos (2007) has explored a more general formulation where there are several fish species involved in a predator-prey relationship. Uncertainty can be added without much complication, as in Koulovatianos (2007).

### 5.4 Differential game models of an oligopolistic fishery

The above fishery models are based on the assumption that individual fishermen have no impact on the market price. This assumption is relaxed in a number of papers, including Dockner et al. (1989), Jorgensen and Yeung (1996), Benchekroun (2003, 2008), Lohoues (2006), Fujiwara (2009a, 2009b). Benchekroun (2003) assumes an inverted V-shaped natural growth function and a linear demand function, and shows that an exogenous unilateral restriction in one firm’s harvest can lead to a decrease in the steady-state stock. In

\[25\] Sandal and Steinshamm (2004) also consider a Cournot oligopoly in a fishery facing a downward sloping linear demand schedule. However, they assume that at most one firm takes into account the stock dynamics. Myopic behavior is also assumed in Hämäläinen et al. (1986, 1990).
Benchekroun (2008), there is an arbitrary number of firms. He shows that an increase in the number of firms results in a lower steady-state industry output. Using an inverted V-shaped natural growth function, Lohoues (2006) shows that equilibrium feedback strategies may exhibit a jump discontinuity with respect to the state variable. Assuming linear dynamics, Fujiwara (2009b) also obtains jumps and shows that steady-state welfare may fall as the number of efficient firms increases if they use feedback strategies, but it increases if they are static Cournot oligopolists. This illustrates the fact that welfare conclusions are very sensitive to the assumption about strategic behavior.

Jorgensen and Yeung (1996) assume that the evolution of the fish stock follows the following stochastic differential equation

\[ dx = \left( ax^{1/2} - bx - \sum_{i=1}^{N} h_i \right) dt + \sigma x dW \quad (53) \]

where \( W \) is a Wiener process, i.e. \( dW \) is normally distributed with mean zero and variance \( \sigma^2 \). Here \( h_i(t) \) is agent \( i \)'s harvest at time \( t \). The parameter \( b \) is the death rate.

Their model can be generalized using a more general function

\[ dx = \left[ ax^\theta - bx - \sum_{i=1}^{N} h_i \right] dt + \sigma x dW \quad (54) \]

Specify \( P = Q^{-(1-\theta)} \) so that the industry’s total revenue is \( PQ = Q^\theta \). The cost function is

\[ C(h_i, x) = \frac{ch_i}{x^{1-\theta}} \]

Without loss of generality, define the harvesting intensity of firm \( i \) as

\[ \omega_i(t) = \frac{h_i(t)}{x(t)} \]

Transform the state variable by defining \( Y = x^\theta \equiv F(x) \). Then

\[ dY = \left[ \theta a - \theta Y \left( b + \frac{1}{2} (1-\theta) \sigma^2 + \sum_{j=1}^{N} \omega_j \right) \right] dt + \theta \sigma Y dW \quad (55) \]

Thus we have transformed the generalized model of Jorgensen and Yeung into a differential game that is linear in the state variable \( Y \). The solution is now straightforward. All players behave identically and use a constant harvesting intensity \( \omega \).

5.5 Entry deterrence

In the models surveyed above, the number of players is exogenously fixed. There are situations where the number of players are endogenously determined, for
example when incumbent firms must choose whether to accommodate or to deter entrants. Crabbe and Long (1993) consider a nation whose fishery industry faces foreign poachers. In the case where poachers take the average catch per vessel as given, the country, acting as the Stackelberg leader, can deter entry poachers by overfishing, as the reduced stock level raises their harvesting costs. In the case where poachers take a more strategic view (i.e., each knows its impact on the marginal product of all vessels), the Stackelberg leader finds it optimal to accommodate entry, and in the steady state there are active poachers. Social welfare of the country decreases in both cases, as compared with sole ownership. Mason and Polasky (1994) consider a two-period model with a single firm facing potential entry of a rival firm. The incumbent deters entry by increasing harvest, thus driving down the resource stock to raise the rival’s harvesting cost.26 They show that social welfare can fall as a result of entry deterrence. There is a parallel between this result and the result on “welfare-reducing enclosure” by Long (1994), who shows that an enclosure decision by private owners of properties, which involves a fixed cost per acre, can reduce welfare.

6 Conclusion and Directions for Future Research

An interesting extension of dynamic games involving transboundary stocks is the study of endogenous coalition formation, under various concepts of coalition stability. The simplest concept of stability is provided by d’Aspremont et al. (1983), according to which stability requires both internal stability (i.e., a member cannot gain from defection, given that all other players maintain their member/non-member status) and external stability (i.e., no outsider wants to become member). This type of stability has been termed “myopic stability” (de Zeeuw, 2008). An alternative concept is the farsighted stability (e.g., Diamond and Sartzetakis 2006). Exploring dynamic games under various possible coalition structures under this concept represents a major challenge.27 Cartel formation and cartel break-up is another interesting issue in the analysis of carbon taxes games. (For cartels, see Benchekroun, Gaudet and Long (2006).)

Another area of research is a dynamic game analysis of side-payments that are conditional on stocks. Long and Sorger (2010) is a first step in this direction. The development of social norms to overcome the tragedy of the commons may also be studied as a differential game, along the line proposed by Benchekroun and Long (2008a).

In specifying dynamic games, authors have typically assumed that all players begin the game at the same time, and have a common time horizon. An interesting area of research is games involving forward-looking overlapping generations.

26The result that renewable-resource firms expand output to deter entry is in sharp contrast with the nonrenewable resource case, where an incumbent would increase its initial price (reducing initial output) in response to threat of a future substitute, leading possibly to lower welfare; see Gilbert and Goldman (1978).

27For some preliminary steps in this direction, see Breton and Keoula (2009) and Rubio and Ulph (2007).
where later generations may or may not care about the welfare of earlier ones. Finally, games among countries having different philosophies have not been explored. In the real world, countries are heterogeneous not only in terms of technology and endowments, but also in terms of philosophical outlook. What happens if one country has the utilitarian objective while the other country has the maximin objective, or a linear combination of the two objectives?

References


The second case is called “ancestor-insensitive welfare function”, a term coined by Asheim (1999). A sketch of such games can be found in Long (2006b).

For infinite-horizon optimization models using a linear combination of the utilitarian objective and the maximin objective, see Alvarez-Cuadrado and Long (2009).


